Time allocation in friendship networks*

Leonie Baumann†

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†Department of Economics, University of Hamburg, 20146 Hamburg, Germany. email: leonie.baumann@wiso.uni-hamburg.de

Abstract

We analyze stable and efficient investments into links (e.g. friendships) and the resulting network structure when agents are endowed with a limited resource (e.g. time) which they can invest into themselves and distribute over all possible links. An agent's utility from a link is produced according to a Cobb-Douglas technology which takes both agents' investments as inputs, and an agent's utility from self-investment follows a strictly concave function. Nash stable networks are either symmetric, "reciprocal", or asymmetric, "non-reciprocal". In the reciprocal equilibrium every agent devotes the same amount of time to herself and two agents' investments into one link match each other. In the non-reciprocal equilibrium the network is bipartite, with a larger set of "unsocial" agents devoting more time to themselves and less time to socializing and a smaller set of "social" agents devoting less time to themselves and more time to socializing. An "unsocial" agent, however, always invests more into a link than her counterpart, a "social" agent. We find that Nash stable regular networks are reciprocal and Nash stable trees are non-reciprocal. There is no pairwise stable component consisting of more than two agents: two agents always have an incentive to team up and spend more time together given the others' investments. The efficient network is reciprocal and features a lower level of self-investment than the Nash stable reciprocal network.

Keywords: network, social networks, link-specific investment, budget constraint, interdependence, time, friendship

JEL Classification Codes: D85, L14, Z13, C72

1 Introduction

Everyone of us has a limited amount of time of 24 hours each day. We spend this time with sleeping, working, doing sports, and meeting friends, among others. Since the time resource is limited, everybody faces the problem of how to "best" spend this time. If we assume agents to be utility maximizers, agents will spend their time in a way which maximizes their utility from time. The allocation of time is the economic problem of how to optimally spend a limited budget.

Within the framework of network theory, this article investigates how much of their limited time utility-maximizing agents devote to their social network - to socializing -, how much they optimally keep for themselves, and how they distribute their "social time" across their social network. We assume that friendships are productive and that the value of each friendship is produced according to a Cobb-Douglas technology that takes both agents' time investments as inputs. This captures the important aspect of social interdependence in friendships: a person's optimal investment into a friendship is most likely dependent on the friend's investment. Time which agents keep for themselves produces a utility for the agent which is strictly concave in "self-investment". We are particularly interested in which time distributions and network structures are Nash, pairwise stable and efficient in this model.

We find that there are two types of Nash stable structures. One type of Nash stable networks are symmetric, reciprocal structures in which every agent chooses the same self-investment and total social investment, and friends match each other's investment into their friendship. The second type of Nash stable networks are asymmetric, non-reciprocal structures which are bipartite with one smaller set of "social and diversified" agents and one larger set of "less social and concentrated" agents. The social and diversified agents choose a lower self-investment and higher total social investment, have more friends on average but always invest less into a friendship than their counterpart, as compared to the larger group of less social but concentrated agents. We highlight the direct social interdependence between agents by introducing a process by which Nash stability can be restored after a Nash stable network has been disturbed. When one friendship between two agents is disturbed, other friendships between agents who are not involved in the initial disturbance are affected and need to be adjusted in their intensity in order to reach a new Nash stable situation.

There is no pairwise stable component of more than two agents because there exist at least two agents who can team up and increase their utility by mutually intensifying their relation and either reducing their investment into another friend unilaterally or into themselves. Efficient networks are symmetric and reciprocal. Yet, they show a lower self-investment than Nash stable reciprocal ones since the positive externality of an agent's social investment on her friends' utilities is accounted for.

This paper contributes to the literature on network formation with endogenous link quality. This literature is embedded in the broader context of research on social and economic networks which has experienced a surge since the mid 90s.The little and rather new work on endogenous link strength can be divided into two strands: articles which treat link investment as specific, i.e. quality, effort or investment levels are specified for each link; and articles which treat link investment as non-specific, i.e. an overall level of link investment, effort, quality is specified, and then affects each link equally.

We will first review the strand treating link investments as specific to which this article belongs, too. Bloch and Dutta (2009) analyze stable and efficient networks when individuals choose a continuous investment into links with others out of a fixed endowment. The authors do not allow for self-investment. For the main part of the analysis they consider an agent's investment to be not directly dependent on its link counterpart's: the value of a link is additively separable in both agents' investments. As an extension, they assume both agents' investments to act as perfect complements.

Brueckner (2006) analyzes friendship networks in which homogeneous individuals enjoy utility from direct and indirect friendships and choose effort levels specific towards other agents to form friendships. Self-investment is not accounted for. Effort requires convex costs but is not resource-constrained, and link formation is stochastic with the probability being an increasing, strictly concave function of each agent's effort level. Though mentioning the possibility of asymmetric solutions for individually as well as socially optimal effort choices, the author concentrates on the characterization of the symmetric solution in which every agent chooses the same effort level towards every other agent.

Also centering their analysis around friendship networks, Tarbush and Teytelboym (2014) model a dynamic process of link formation where every agent spends the same exogenously given proportion of time with a social group. They examine the dynamics of the system which do not rely on optimizing behavior of agents but on a stochastic process solved by mean field approximation.

Rogers (2006) analyzes Nash equilibrium and efficient investments of agents into specific links to others. Self-investment is not allowed for. Investments are subject to an individual resource constraint and directly benefit only one side of the link. Either an agent's own or the counterpart's utility are directly affected. This is contrary to our model in which each agent's utility from a link is a function of both parties' investments. In Rogers (2006) an agent's utility (value) is the sum of her intrinsic value and the utility (value) of her directly linked agents weighted by the corresponding link investments. Hence, agents' values are endogenous and reinforce each other.

Looking at R&D networks, Goyal, Moraga-González and Konovalov (2008) analyze a firm's choice of investment in in-house research and in each research project with another firm. Research reduces costs and firms compete in the market. Firms choose investment levels that maximize their profits whereas in the present paper agents allocate a given resource in a utility-maximizing way. The authors focus on symmetric investment levels in equilibrium, i.e. every firm invests the same into each project with another firm and every firm invests the same into in-house research.

Next, we give an overview of the other, not as closely related strand of literature. These are models of non-specific networking in which agents choose an overall investment into their social network, but not link-specific investments.

Golub and Livne (2010) investigate the equilibrium non-specific effort choices of agents to form links to other agents when utility is derived from direct and indirect links. Link formation is stochastic with the probability being a strictly increasing function of both agents' efforts. Effort requires costs which have a convex form and agents are heterogeneous with respect to a cost parameter. A resource constraint is not considered. Also, self-investment is not accounted for. They focus on symmetric equilibria in which strategy choice only depends on the cost type and not on the individual, i.e. same types behave in the same way. The authors show that if they allow agents to choose a specific level of effort for other agents, i.e. to discriminate between agents, choosing the same effort level towards every other agent is equilibrium if agents incur a cost for discriminating. This corresponds to a non-specific effort level which is uniformly distributed over others and lets the authors conclude that this assumption in the baseline model is not strict.

Cabrales, Calvó-Armengol and Zenou (2011) determine an agent's choice of a costly productive effort and of a costly non-specific socializing effort which is distributed over links proportionate to the other agent's social effort. Effort levels are not constrained by a limited resource. Galeotti and Merlino (forthcoming) analyze an agent's overall investment into a job-contact network, and Durieu, Haller and Solal (2011) examine the Nash equilibria of a large class of networking games in which agents choose non-specific networking efforts.

This paper is the first, to the author's best knowledge, to analyze the combination of utility-maximizing link-specific investments which feature direct social interdependence and influence both agents' utilities, and self-investment subject to a resource constraint. Moreover, we do not focus on symmetric solutions but treat symmetric and asymmetric situations equally, and are able to derive, next to symmetric solutions, interesting results on asymmetric solutions, as already hinted to above.

The structure of the paper is as follows. In section 2 we introduce the model. Then, Nash stable and pairwise stable time distributions and network structures are characterized and discussed in section 3. Section 5 compares agents' utilities in the different stable networks and presents the welfare maximizing time distribution and network structures. Section 6 concludes.

2 Model

The social network consists of a set of agents $N = \{1, ..., n\}$. Each agent i possesses the amount of time $T > 0$ which she can allocate over the social network and herself. We denote her time "investment" into the relationship with player $j \neq i$ as t_{ij} and the time she keeps for herself as t_{ii}. We will occasionally refer to the time she keeps for herself as "self-investment", this can be time for working or sleeping, for example. By nature time investments are constrained: $t_{ij} \geq 0$ for all j and $\sum_j t_{ij} \leq T$. An agent's time investment strategy is the vector $t_i = (t_{i1}, ..., t_{in})$.

The network will be completely described by the matrix of time investment strategies

$$\mathscr{T} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{pmatrix}.$$

We define a link (friendship) ij between two agents i and j to exist if both $t_{ij} > 0$ and $t_{ji} > 0$. Both i and j must invest a positive amount of time into each other for a direct link to form. The social network is undirected in terms of links since no player i can have a link to player j if player j is not linked to player i, i.e. if link ij exists, then also link ji exists (Jackson, 2005). Note this only applies to link existence but does not imply that agents i and j invest the same amount into each other, i.e. that $t_{ij} = t_{ji}$.

The value of friendship ij for agent i depends on the time both agents i and j invest. Intuitively, a friendship gets stronger if time investments (equivalent to effort in friendship) are increased. However, the positive effect of a unilateral increase on the value of friendship is likely to be decaying. Moreover, a friend's investment will be dependent on the other's investment: An agent is likely to be willing to invest more into a friendship if she observes that the other is exerting a higher effort. To capture these properties of friendship creation (production), we resort to the Cobb-Douglas technology and, more specifically, assume that the value from friendship ij for agent i

is the function

$$v_{ij}(t_{ij}, t_{ji}) = a_j \, t_{ij}^{\beta_i} \, t_{ji}^{1-\beta_i}.$$

Positive, but diminishing partial derivatives and positive cross-derivatives for positive investment levels reflect the above intuition. a_j can be seen as the intrinsic value of agent j for all other agents, thus, is independent of i. The assumption that agent j offers the same intrinsic value to all other agents implies that all agents agree upon the intrinsic value of agent j. This seems justified in the context of educational level, for example, but is not true for every case, as for example humor is perceived in different ways. $\beta_i \in (0, 1)$ is the value elasticity of agent i's time investment and measures the complementarity of agent i's and j's time investment in the value creation for agent i. β_i is agent i specific and independent of j. Assuming that both exponents sum up to unity, implies constant returns to scale to both agents' investments. This seems to be the least strict assumption, if one is to make an assumption on returns to scale. Assuming decreasing returns to scale would bias our efficiency results towards agents diversifying their time investment over friendships, increasing returns to scale would bias our efficiency results towards agents concentrating their time investment, whereas constant returns to scale do not push our efficiency results into one direction a priori, but rather leave any direction possible.

Furthermore, note that if $t_{ij} = 0$ and/or $t_{ji} = 0$ and hence the link ij does not exsit, $v_{ij}(t_{ij}, t_{ji}) = 0$. Moreover, the marginal value from friendship is zero for a unilateral increase in time investment by agent i (j), if j (i) does not invest. Thus, a friendship cannot be established and friendship benefits cannot be enjoyed without some consent of the counterpart.

Agent i's utility from self-investment is assumed to be given by the increasing, strictly concave and differentiable function $f_i(t_{ii})$. Moreover, $f_i'(t_{ii}) \to \infty$ if $t_{ii} \to 0$. The latter assumption is natural as, for example, an agent needs some time for sleeping. Heterogeneity in f_i could serve to model heterogeneity in time constraints for socializing. Compare for example two students, one gets financed by her parents, the other needs to earn her own money, and who are the same in every other aspect. Then the not-financially supported student will have a higher marginal utility for every amount of self-investment compared to the financially supported one, since she needs to finance herself with her private time.

Let us highlight that both agent i's utility from a friendship ij and her utility from self-investment are strictly concave in t_{ij} and t_{ii}, respectively. Hence, an individual agent's time always features decreasing marginal returns. The joint investment of two agents has a higher productivity: decreasing marginal returns of the two individual investments add up to constant marginal returns of joint investment in friendship production as follows from the exponents in $v_{ij}(t_{ij}, t_{ji})$.

The total utility of agent i from network $\mathscr{T}$ is defined to be the sum of her utility from her social network – the sum of all friendship values – and her utility from self-investment:

$$u_i(\mathscr{T}) = \sum_{j \neq i} a_j t_{ij}^{\beta_i} t_{ji}^{1-\beta_i} + f_i(t_{ii}).$$

Agents are utility maximizers. Agent i chooses her best response t_i^* subject to the constraints on her time investment.

3 Nash stable networks

3.1 Time allocations

To examine stable time allocations and network structures, we use the concept of Nash stability after Bloch and Dutta (2009):

Definition 1. *A network $\mathscr{T}$ is Nash stable if there exists no agent i who is strictly better off by unilaterally deviating from investment strategy t_i to another feasible investment strategy t_i'.*

A Nash stable network $\mathscr{T}$ is the Nash equilibrium of the model. Thus, in order to determine the Nash stable networks, we derive agent i's best response t_i^*. If agent j does not invest into i, then it is also optimal for i not to invest time into j because no friendship between i and j exists and i

cannot derive positive utility from investing in j. If agent j does invest into i, then it is also optimal for agent i to invest into j, because for $t_{ij} \to 0$ the marginal utility from friendship ij approaches infinity. Similarly, it is optimal for agent i to always invest into herself since for $t_{ii} \to 0$ marginal utility from self-investment approaches infinity. Hence, we summarize, $t^*_{ij} = 0$ if $t_{ji} = 0$, $t^*_{ij} > 0$ if $t_{ji} > 0$, and $t^*_{ii} > 0$. Moreover, t^*_i is such that she allocates her whole endowment T as spending time is always utility-enhancing.

Knowing that $t^*_{ik} = 0$ for all $t_{ki} = 0$, the agent's time allocation problem reduces to choosing the optimal level of time investment, t^*_{ij}, into each friendship ij for which $t_{ji} > 0$ and the optimal level of self-investment, t^*_{ii}. From the Lagrangian and after rearranging, we derive the following First Order Conditions for every agent i for the utility maximizing level of all t^*_{ij} and t^*_{ii}:

$$a_j \beta_i \left(\frac{t_{ji}}{t_{ij}}\right)^{1-\beta_i} = f'_i(t_{ii}) \quad \forall \quad j \neq i \tag{1}$$

$$\sum_j t_{ij} = T \tag{2}$$

Thus, the optimal investment strategy t^*_i for agent i is to choose $t^*_{ij} > 0$ for $t_{ji} > 0$ and $t^*_{ii} > 0$ such that Equations 1 and 2 are fulfilled, and to choose $t^*_{ik} = 0$ for all $t_{ki} = 0$. From Equation 1 we can see that the optimal investment strategy of agent i equates the marginal utilities from each of her existing friendships to her marginal utility from self-investment. Equation 2 captures that the budget constraint is binding. These results are captured in Proposition 3.1.

Proposition 3.1. *The network $\mathscr{T}$ is Nash stable if and only if each agent i chooses t^*_i such that*

(a) (2) holds,

(b) $t^*_{ij} = 0$ for all j for which $t^*_{ji} = 0$,

(c) $t^*_{ii} > 0$ and $t^*_{ij} > 0$ for all j for which $t^*_{ji} > 0$ such that (1) is satisfied.

Having denoted best responses with superscript *, we allow ourselves some sloppiness in notation and denote investments strategies and all other values in a Nash stable network with * from now on.

From (1) we can see that self-investment and investment into a friendship as well as friendship investments among themselves are complementary because

$$\frac{dt_{ij}}{dt_{ii}} = \frac{f_i''(t_{ii})}{a_j\beta_i(\beta_i - 1)t_{ji}^{1-\beta_i}t_{ij}^{\beta_i-2}} > 0.$$

In a Nash stable network, a higher level of self-investment of agent i also implies a higher level of investment of agent i into each of her existing friendships. Yet, we need to keep in mind that overall investment is bounded from above by the budget constraint.

From Proposition 3.1 follows that the empty network with $t_{ij}^* = 0$ and $t_{ii}^* = T$ for all agents i is Nash stable. This proves existence of a Nash stable network. The empty network is a strict Nash equilibrium, in the sense that every agent is choosing a unique best response.

Next, we derive further characteristics about the time investment structure of Nash stable networks different from the empty network. For this we first introduce some necessary definitions. A path between i and j is "a sequence of links $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ such that" for each i_ki_{k+1} with $k \in \{1, ..., K-1\}$, $i_1 = i$ and $i_K = j$, $t_{i_ki_{k+1}} > 0$ and $t_{i_{k+1}i_k} > 0$, i.e. the link between i_k and i_{k+1} exists, and such that no node appears twice in the sequence $i_1, ..., i_K$ (Jackson, 2008, p. 23). Whenever we refer to "length", we mean the number of links involved. We say two are agents i and j are connected if there exists a path between i and j (Jackson, 2008, p. 26). A component of a network is a non-empty subset of agents who are connected among themselves, i.e. there exists a path between any two agents in the component, but who are not connected to any agent outside the component (Jackson, 2008, p. 26). We say a component C consisting of the set of agents $N^C \subseteq N$ is Nash stable if every $i \in C$ chooses an optimal investment strategy t_i^{C*} towards all other agents $j \in C$.

Lemma 3.2. *A network $\mathscr{T}$ is Nash stable if and only if it consists of Nash* *stable components.*

For the first part of Lemma 3.2, observe that in the Nash stable component C every agent $i \in C$ chooses t_i^{C*}. Now take all Nash stable components of network $\mathscr{T}$, then an optimal strategy of agent $i \in C$ towards the whole network is t_i^{C*} and $t_{ik}^* = 0$ towards all agents $k \notin C$. Agents from other components do not invest into i, so i's best response is not investing. Hence, a network $\mathscr{T}$ consisting of Nash stable components is Nash stable. For the second part of Lemma 3.2, take a Nash stable network where agent i's optimal strategy is t_i^*. The optimal time investment towards all agents $k \notin C$ is $t_{ik}^* = 0$. Then the time investments in t_i^* towards all agents $j \in C$ are also an optimal investment strategy for agent i if she chooses an optimal investment strategy towards agents in C only. We only neglect zero investments ($t_{ik}^* = 0$) of t_i^* so the budget constraint is still binding and marginal utilities are still equalized. Hence, time investments towards all agents $j \in C$ of t_i^* constitute a t_i^{C*}. So if a network is Nash stable then also all its components are Nash stable. Thus, in the following we only need to determine properties of Nash stable components to describe Nash stable networks.

From now on, we restrict our attention to homogenous agents where $f_i = f$, $\beta_i = \beta$, and $a_i = a$ for all i.

Lemma 3.3. *In every Nash stable component,*

agent i allocates time to friends j and k such that the ratios of time investments $\frac{t_{ji}^*}{t_{ij}^*}$ and $\frac{t_{ki}^*}{t_{ik}^*}$ are equal, i.e. $\frac{t_{ji}^*}{t_{ij}^*} = \frac{t_{ki}^*}{t_{ik}^*} := r$.

Agent j (k) faces the reverse ratio of time investments in all of her friendships jl (kl), i.e. $\frac{t_{lj}^*}{t_{jl}^*} = \frac{1}{r}$ ($\frac{t_{lk}^*}{t_{kl}^*} = \frac{1}{r}$).

Proof in appendix.

The result that from agent i's perspective ratios of time investments are equal to r in all of her friendships derives from the equation of marginal utilities in equilibrium. That agent j (k), each friend of agent i, faces the reverse ratio of time investment $1/r$ in all of her friendships is due to the direct social interdependence of i's and j (k)'s time investment in their friendship.

All agents in a component are connected. Hence, in a Nash stable component for every agent $i \in C$ the ratio of time investments from all of her friendships is either equal to r or $1/r$ from the perspective of the agent and a Nash stable time distribution features exactly one r. From here on, we will denote with r the ratio which is greater or equal to one, $r \geq 1$. If the ratio of time investments $r = \frac{t_{ji}}{t_{ij}}$ is equal to 1, then friends i and j invest the same amount of time into each other. We will call this symmetric situation "reciprocal". If $r > 1$ then i invests more into each friend j than each j into i, and j invests less into each of her friends (among which is also i) than her friends into j. We will call this asymmetric situation "non-reciprocal". Because a time distribution inside a Nash stable component determines exactly one r, a Nash stable component is either reciprocal or non-reciprocal.

Lemma 3.4. *In a Nash stable component C which is*

***reciprocal** ($r = 1$)**,** every agent $i \in C$ chooses the same self-investment t^*_{ii} such that $a\beta = f'(t^*_{ii})$, and the same total time for socializing $TS^*_i = T - t^*_{ii}$.*

***non-reciprocal** ($r > 1$)**,** there exist two levels of self-investment and of time for socializing in C, with 2 friends, k and l in C, choosing different levels from each other.*

With $\frac{t_{lk}}{t_{kl}} = r$,

each k chooses self-investment t^*_{kk} such that $a\beta r^{1-\beta} = f'(t^*_{kk})$ and total time for socializing $TS^*_k = T - t^*_{kk}$,

each l chooses self-investment t^*_{ll} such that $a\beta \left(\frac{1}{r}\right)^{1-\beta} = f'(t^*_{ll})$ and total time on socializing $TS^*_l = T - t^*_{ll}$, and invests more into a friendship with k than k does.

The ordering of self-investments is $t^*_{kk} < t^*_{ii} < t^*_{ll}$ and hence of total social time $TS^*_l < TS^*_i < TS^*_k$.

Proof. Lemma 3.4 follows from Lemma 3.3 and from substituting r into the FOCs. Since the budget constraint is binding the total social time for each agent is the difference between her budget and her self-investment. The ordering of investments is implied by the curvature of f. $\square$

In a Nash stable reciprocal component agents are to some degree symmetric: every agent invests the same amount into a friendship as her friend, every agent chooses the same level of self-investment and hence the same level of total social investment. Yet, friendships can be of different strength. Links can feature a low or high investment. Moreover, agents can have a different number of friends.

In a Nash stable non-reciprocal component agents are more asymmetric. There are two types of agents k and l with friends being of different type. Compared to the reciprocal case, type k chooses lower self-investment and hence higher total social time, and type l chooses higher self-investment and hence less total social time. Type l always invests more into a friendship with k than k does.

3.2 Component structures

Until here we have concentrated on the properties of the equilibrium time investment strategies of agents. In the following we will analyze which component structures in terms of link existence are underlying the different Nash stable time distributions. For the following, we define a cycle. A cycle is a sequence of links $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ where $i_1 = i_K$ and "$i_k \neq i_{k'}$ for $k < k'$ unless $k = 1$ and $k' = K$" (Jackson, 2008, p. 24). Hence a cycle is a path except for having the same start and end node.

Lemma 3.5. *A component containing a cycle of odd length can only be Nash stable if agents in the component reciprocate, i.e. $t^*_{ij} = t^*_{ji}$ for all $i, j \in C$.*

Lemma 3.5 is explained by a graphical example. Figure 1 shows an odd cycle of agents 1, 2 and 3 which might be part of a larger component. From Lemma 3.3 we know that in the Nash stable component it must be true that $\frac{t^*_{21}}{t^*_{12}} = \frac{t^*_{31}}{t^*_{13}} = r$ for agent 1 (w.l.o.g. $= r$ and not $= 1/r$), $\frac{t^*_{12}}{t^*_{21}} = \frac{t^*_{32}}{t^*_{23}} = \frac{1}{r}$ for agent 2 , and $\frac{t^*_{23}}{t^*_{32}} = r = \frac{t^*_{13}}{t^*_{31}} = \frac{1}{r}$ for agent 3. For the FOCs to be fulfilled for agent 3, r must be equal to unity. Hence, with an odd cycle time investments into a friendship must match each other, otherwise the system of FOCs cannot be solved. Though, we only present an example with three agents here, the same reasoning is easily extended to a cycle of arbitrary odd length. In the appendix we provide a proof for any cycle of odd length.

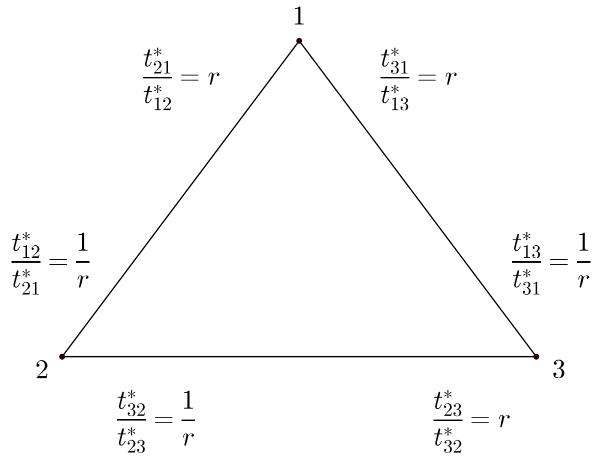

Figure 1: Example for Lemma 3.5.

For Lemma 3.6 we define a leaf. A leaf is an agent who has only one link (Jackson, 2008, p. 27).

Lemma 3.6. *A component with $n^C \geq 3$ containing a leaf can only be Nash stable if the component is non-reciprocal.*

In a Nash stable component with $n^C \geq 3$ each leaf is of type l choosing a higher level of self-investment and investing more into her only friendship than her friend who is of type k.

To see why a leaf must be of type l in a Nash stable component with $n^C \geq 3$, consider the following:

Proof. Without loss of generality, let agent 1 be a leaf and agent 2 be the only friend of 1. As $n^C \geq 3$, agent 2 is not a leaf, so has more friends than only agent 1. We will show by contradiction that agent 1 must be type l if the component is Nash stable. Suppose in Nash equilibrium agent 1 is not type l such that $t^*_{12} \leq t^*_{21}$. Then,

$$a\beta \left(\frac{t^*_{21}}{t^*_{12}}\right)^{1-\beta} = f'(t^*_{11}) \geq a\beta \left(\frac{t^*_{12}}{t^*_{21}}\right)^{1-\beta} = f'(t^*_{22}).$$

The left hand side of the inequality sign is part of agent 1's FOCs whereas the right hand side is part of agent 2's FOCs. Thus, because of the strict concavity of f it follows that $t^*_{11} \leq t^*_{22}$. As $t^*_{11} + t^*_{12} = T \Rightarrow t^*_{22} + t^*_{21} \geq T$. This is a contradiction. In case of inequality the endowment constraint of 2 is violated, and in case of equality it shows that 2 only has one friend which violates our assumption. Thus, in Nash equilibrium $t^*_{12} > t^*_{21}$ and

$$a\beta \left(\frac{t^*_{21}}{t^*_{12}}\right)^{1-\beta} = f'(t^*_{11}) < a\beta \left(\frac{t^*_{12}}{t^*_{21}}\right)^{1-\beta} = f'(t^*_{22}) \Rightarrow t^*_{11} > t^*_{22}.$$

Hence, the component is non-reciprocal and agent 1 is type l. □

The only Nash stable component which is reciprocal and contains leaves is the component with $n^C = 2$. Indeed, the only possibility to form a component of $n^C = 2$ is to connect both agents and then each of them is a leaf. Such a component can only be Nash stable if it is reciprocal. The proof can be constructed in a similar way as the proof of Lemma 3.6 and will be omitted since we do not focus on the special case of $n^C = 2$.

Corollary 3.7. *No component of $n^C \geq 3$ with an odd length path between two leaves is Nash stable.*

No component with both an odd cycle and a leaf is Nash stable.

The first statement of Corollary 3.7 derives from combining Lemmata 3.4 and 3.6. From Lemma 3.6 we know that in a Nash stable component a leaf must be of type l and Lemma 3.4 implies that being of type l and k alternates between linked agents. If there is an odd length path between two leaves and one of the leaves is of type l, then because of the alternation of types between friends, the other leaf would be of type k which is not Nash stable. The second statement is obtained by combining Lemmata 3.5 and 3.6. In a Nash stable component an odd cycle requires reciprocity and a leaf non-reciprocity. Since a Nash stable component is either reciprocal or non-reciprocal, a leaf and an odd cycle cannot coexist in a Nash stable component.

A component is a tree if it has no cycle (Jackson, 2008, p. 27). From Lemmata 3.6 and 3.7 we can derive the following result about trees because a tree has at least two leaves.

Corollary 3.8. *A tree can only be Nash stable if it is non-reciprocal, if there is no odd length path between two leaves and if leaves are of type l.*

For the following we define a bipartite component. A component C is bipartite if N^C can be partitioned into two disjoint sets A and B such that $A \cup B = N^C$ and such that if there exists a link between two nodes then one node belongs to set A and the other one to set B (c.f. Jackson, 2008, p. 43). So there exist only links across the sets and not within one set.

If a Nash stable component C is non-reciprocal, then the component is bipartite. This follows from Lemma 3.4. Friends choose different levels of self-investment and total time for socializing, one is of type k and the other one is of type l. Let us denote the set of nodes that contains all $l \in C$ as L^C, and let us denote the other set of nodes that contains all $k \in C$ as K^C.

If a Nash stable component C is reciprocal, the component can be non-bipartite or bipartite. An example for a Nash stable component which is reciprocal and bipartite is a circle with an even number of nodes. An example for a Nash stable component which is reciprocal and non-bipartite is every component which has an odd length cycle. We will stick to denote the two disjoint sets as L^C and K^C in the case of a bipartite Nash stable component which is reciprocal.

We exploit the concept of bipartiteness and properties of the equilibrium investment strategies of k and l to derive properties of the cardinality of K^C and L^C in Nash stable bipartite components. Since both $r = 1$ and $r > 1$ can imply bipartiteness of a Nash stable component we work with $r \geq 1$.

Table 1 shows properties of the investment strategies of each $l \in L^C$ and each $k \in K^C$ in a bipartite Nash stable component. TS_i denotes the total amount of time agent i spends on socializing. From Table 1 follows that the sum of time investment that all l devote to friendships is at least as large

$$r = \frac{t_{lk}}{t_{kl}} \geq 1$$

$\forall\ l \in L^C$		$\forall\ k \in K^C$
t_{ll}	$\geq$	t_{kk}
$TS_l := T - t_{ll}$	$\leq$	$TS_k := T - t_{kk}$

Table 1: Properties of investment strategies in Nash stable bipartite components.

as the sum of time investment that all k devote to friendships: for every friendship kl it is true that $t_{lk} \geq t_{kl}$, and hence $\sum_l \sum_k t_{lk} \geq \sum_k \sum_l t_{kl}$. As $TS_i = \sum_{j \neq i} t_{ij}$, $\sum_l TS_l \geq \sum_k TS_k$. Yet, we know that each l spends at most as much on socializing as each k. Thus, it must be true that the number of l agents must be at least as large as the number of k agents: $\left|L^C\right| \geq \left|K^C\right|$ because $TS_l \leq TS_k$. From this we derive Proposition 3.9.

Proposition 3.9. *If a Nash stable component is bipartite and reciprocal, the number of agents in L^C and K^C is the same: $\left|L^C\right| = \left|K^C\right|$.*

If a Nash stable component is non-reciprocal, the number of agents who invest more into each friendship than their friend, more into themselves, and less into socializing is larger than the number of agents who invest less into each friendship than their friend, less into themselves and more into socializing: $\left|L^C\right| > \left|K^C\right|$.

Due to bipartiteness, the sum of friendship links emanating from all $k \in K^C$ is the same as the sum of friendship links emanating from all $l \in L^C$. This implies that for $r = 1$ and hence $\left|L^C\right| = \left|K^C\right|$, l and k have the same number of friends on average. For $r > 1$ and hence $\left|L^C\right| > \left|K^C\right|$, k has on average more friends than l. Our results lead to Proposition 3.10 for which we define a regular component. A regular component is a component in which all nodes have the same number of links (Jackson, 2008, p. 30). We will denote a regular component in which every i has d friends as a d-regular component.

Proposition 3.10. *A regular component is Nash stable if and only if it is reciprocal, $a\beta = f(t^*_{ii})$ and $\sum_j t^*_{ij} = T$ for all i.*

Proof. If any regular component is reciprocal and $a\beta = f(t^*_{ii})$ and $\sum_j t^*_{ij} = T$ for all i then the FOCs are fulfilled and the component is Nash stable.

If a regular component is Nash stable then every agent $i \in C$ exhausts her budget and chooses t^*_{ii} such that $a\beta \left(\frac{1}{r}\right)^{1-\beta} = f(t^*_{ii})$ or $a\beta r^{1-\beta} = f(t^*_{ii})$. This follows from the FOCs and from Lemma 3.4. Hence, $\sum_j t^*_{ij} = T$ for all i. To show that every regular Nash stable component is reciprocal assume the contrary. Assume that $r \neq 1$. Then the component is bipartite with two sets L^C and K^C and each $k \in K^C$ has on average more friends than each $l \in L^C$ (cf. Proposition 3.9). But then the component is not regular which is a contradiction. Thus, every Nash stable regular component must be reciprocal with $r = 1$. $\square$

In a Nash stable regular component each agent $i \in N^C$ devotes the same amount of time TS^*_i to socializing and t^*_{ii} to self-investment because $r = 1$ (cf. Lemma 3.4). Special cases of regular components are the circle, in which every $i \in N^C$ has exactly two links, and the complete component, in which every $i \in N^C$ is linked to every other $j \in N^C$ with $j \neq i$. The most simple Nash stable d-regular component is the symmetric one, in which every i invests $t^*_{ij} = \frac{TS^*_i}{d}$ into every friend j and $t^*_{ii} = T - TS^*_i$ into herself. Yet, equal investments in all links are not necessary for a regular component to be Nash stable. Many Nash stable regular components allow agent i to have "good" friends g with $t^*_{ig} > \frac{TS^*_i}{d}$, and "bad" friends b with $t^*_{ib} < \frac{TS^*_i}{d}$ as long as both are balanced, i.e. as long as $\sum_{j \neq i} t^*_{ij} = TS^*_i$.

With the help of Theorem 35.1 by Schrijver (2004, p. 584) we characterize the complete set of component structures for which a reciprocal Nash stable distribution of time investments exists. A component structure is defined by the set of nodes in the component and the set of existing links among these nodes. The time investment on the existing links is not assigned. Let $G^C = (N^C, E^C)$ describe the component structure, with N^C being the set of nodes in component C and E^C being the set of links that exist among N^C. Further, let $U \subseteq N^C$ and $G^C - U$ be the component structure that results after deleting the set of nodes U from G^C: $G^C - U = \left(N^C \backslash U, E^C \backslash \{ij \mid i \in U\}\right)$. Hence,

$G^C - U$ contains all nodes $N^C \backslash U$ and all links of E^C that do not involve any $i \in U$. Denote by W the set of isolated nodes in $G^C - U$ with an isolated node being a node without any links.

Theorem 3.11. *There exists a Nash stable reciprocal distribution of time investments for $G^C = (N^C, E^C)$ with $t^*_{ij} > 0$ if $ij \in E^C$, and $t^*_{ij} = 0$ if $ij \notin E^C$ if and only if for every $U \subseteq N^C$ with W being the set of isolates of $G^C - U$ either*

1. $|U| > |W|$,

or

2. $|U| = |W|$ *and for every link $ij \in E^C$ with $i \in U$ it is true that $j \in W$.*

The proof how Theorem 3.11 follows from Schrijver (2004, p. 584) is relegated to the Appendix.[1]

Thus, according to Theorem 3.11 there exists a Nash stable reciprocal distribution of time investments if and only if either 1. the number of nodes in U is larger than the number of isolates W in $G^C - U$, or 2. the number of nodes in U is the same as the number of isolates W in $G^C - U$ and every $i \in U$ is only linked to $j \in W$ in G^C.

The intuition behind Theorem 3.11 is easily explained. Deleting a set of nodes U from a component structure can leave some nodes isolated (W) because in the original component they are only linked to U. In a Nash stable reciprocal solution for G^C the nodes in W would require an overall social time investment of $|W|\, TS^*_i$ from U (because of reciprocity). If $|U| < |W|$, the overall social time nodes in U can give to W, $|U|\, TS^*_i$, is strictly less than the nodes in W require. Thus, a Nash stable reciprocal time distribution never exists in this case. If $|U| = |W|$, the overall social time nodes in U can give to W, $|U|\, TS^*_i$, is exactly as the nodes in W require. Hence, if nodes in U spend their social time on nodes in W only (every node in U is only linked to nodes in W), then a Nash stable reciprocal time distribution exists. If $|U| > |W|$, the overall social time nodes in U can give to W,

[1]Theorem 3.11 came to life with the great help of Henning Bruhn-Fujimoto.

$|U|\, TS_i^*$, is larger than the nodes in W require. Then there exists a Nash stable reciprocal time distribution in which nodes in U spend $|W|\, TS_i^*$ on nodes in W. The nodes in U invest their remaining social time on other nodes in U or nodes in $N^C \setminus \{U \cup W\}$ in some way. There is no restriction on the existence of links towards other nodes in U or in $N^C \setminus \{U \cup W\}$ since the required minimal investment in some link is only $\epsilon > 0$.

To illustrate the application of Theorem 3.11 we briefly derive two of our previous results. We showed that a leaf can never exist in a Nash stable reciprocal component with $n^C \geq 3$. If we take $U = \{\textit{the only friend of the leaf}\}$, then $W = \{\textit{the leaf}\}$ such that every G^C containing a leaf violates 1. and 2. of Theorem 3.11. Moreover, we can retrieve a result close to Proposition 3.9: No bipartite component with $\left|L^C\right| \neq \left|K^C\right|$ has a Nash stable reciprocal solution. Taking the smaller set of $\left|L^C\right|$ and $\left|K^C\right|$ as U always results into $|U| < |W|$.

Theorem 3.11 provides a new insight into the Nash stability of component structures in which every node is friends with more than half of all other nodes in the component.

Corollary 3.12. *For every component structure in which each $i \in N^C$ is friends with more than half of the component population a Nash stable reciprocal time distribution exists.*

If the number of friends of each $i \in N^C$ is greater than $\frac{n^C}{2}$ then more than $\frac{n^C}{2}$ nodes have to be deleted in order to have at least one isolate in $G^C - U$. For $|U| \leq \frac{n^C}{2}$, $|W| = 0$, and for $|U| > \frac{n^C}{2}$, $|W| < |U|$ because $\left|G^C - U\right| < |U|$. So for each $U \subseteq N^C$, $|U| > |W|$.

3.3 Social Interdependence

In the following we will discuss how we can "move" between Nash stable components. Moving from one Nash stable component to another requires shifting time investments of agents. We define a process which allows us to reach a new Nash stable component from a given Nash stable component with both components featuring the same r. The only things affected by the process are link intensities and potentially link existence. The process can be applied to reciprocal as well as non-reciprocal Nash stable components. This section sheds more light on the direct social interdependency in the network, namely how agents are affected by changes in links between others. Moreover, we see which adjustments need to happen in order to reach a new Nash stable situation after some Nash stable component has been disturbed.

Let C with $\mathscr{T}^C$ be a Nash stable component with r^C or a number of Nash stable components which share the same r^C. Take an even length sequence of agents $S = (i_1 i_2, i_2 i_3, ..., i_M i_{M+1})$ with $M > 2$ from $i \in C$ such that $i_m \neq i_{m+1}$ and $i_1 = i_{M+1}$. If $r^C > 1$ then additionally i_m and i_{m+1} must be of different types, hence, either i_m is type k and i_{m+1} is type l or vice versa. We do not require the link $i_m i_{m+1}$ to exist in $\mathscr{T}^C$.

Then C' with $\mathscr{T}^{C'} \neq \mathscr{T}^C$ and $r^{C'} = r^C$ which results after changing the time investment from i_m into i_{m+1} by $\Delta_{i_m i_{m+1}}$ and from i_{m+1} into i_m by $\Delta_{i_{m+1} i_m}$ is Nash stable if

1. $t'_{ij} = t^*_{ij} + \sum_{i_m i_{m+1} = ij} \Delta_{i_m i_{m+1}} \geq 0$,
2. $t'_{ji} = t^*_{ji} + \sum_{i_{m+1} i_m = ji} \Delta_{i_{m+1} i_m} \geq 0$,
3. for $r^C = 1$, $\frac{\Delta_{i_{m+1} i_m}}{\Delta_{i_m i_{m+1}}} = 1$;

   for $r^C > 1$, $\frac{\Delta_{i_{m+1} i_m}}{\Delta_{i_m i_{m+1}}} = r^C$ for all m of type k,

   and hence $\frac{\Delta_{i_{m+1} i_m}}{\Delta_{i_m i_{m+1}}} = \frac{1}{r^C}$ for all m of type l,
4. $\Delta_{i_m i_{m+1}} = -\Delta_{i_m i_{m-1}}$,
5. and $t'_{ij} = t^*_{ij}$ for $ij \notin S$.

1. and 2. ensure that time investments in $\mathscr{T}^{C'}$ are again non-negative. 3. ensures that $r^{C'} = r^C$ and thus that marginal utilities are also equated in $\mathscr{T}^{C'}$, and 4. ensures that every agent's budget is again exhausted in $\mathscr{T}^{C'}$.

Some comments are in order to give more insight into the described process. Note that S is a cycle that can contain agent i multiple times and a pair of agents in S is not restricted to two agents who are friends but can also consist of two agents who are not friends.

Furthermore, there exist at most two different absolute values of change in time investment in this process of time shifting, namely, $\Delta_{i_1 i_2} = \Delta_{i_{2p+1} i_{2p+2}}$ and $\Delta_{i_2 i_1} = \Delta_{i_{2p+2} i_{2p+1}}$ for $p \in \left\{1, ..., \frac{M}{2} - 1\right\}$ where $\Delta_{i_1 i_2} = \Delta_{i_2 i_1}$ if $r^C = 1$ and $\Delta_{i_1 i_2} \neq \Delta_{i_2 i_1}$ if $r^C > 1$:

From 3. we can infer that

$$\frac{\Delta_{i_{m-2} i_{m-1}}}{\Delta_{i_{m-1} i_{m-2}}} = \frac{\Delta_{i_m i_{m-1}}}{\Delta_{i_{m-1} i_m}} = \frac{\Delta_{i_m i_{m+1}}}{\Delta_{i_{m+1} i_m}},$$

and from 4.

$$\frac{\Delta_{i_{m-2} i_{m-1}}}{\Delta_{i_{m-1} i_{m-2}}} = \frac{-\Delta_{i_m i_{m+1}}}{-\Delta_{i_{m-1} i_{m-2}}} = \frac{\Delta_{i_m i_{m+1}}}{\Delta_{i_{m+1} i_m}}.$$

Hence, $\Delta_{i_{m-2} i_{m-1}} = \Delta_{i_m i_{m+1}}$ and $\Delta_{i_{m-1} i_{m-2}} = \Delta_{i_{m+1} i_m}$.

By the process, friendships can be created, dissolved or only changed in their intensity. There exist time shifting processes that merge multiple Nash stable components into one Nash stable component and that decompose one Nash stable component into multiple Nash stable components.

We give three examples with Figure 2, 3, and 4. In these examples, nodes are numbered in black. Time investments on a link between node i and j are indicated in red. In reciprocal Nash stable components time investments of node i and j into their link are the same, so are characterized by one investment t. In non-reciprocal Nash stable components time investments of i and j into their link are not the same and are put down separately with t_{ij} meaning that i invests t into j. The figure title of each example indicates r^C characteristic for the Nash stable component C and the self-investment which is assumed to be the unique solution to the system of FOCs given specific functional forms and r^C. This is without loss of generality for the examples. On the left hand side of each example we depict C with $\mathscr{T}^C$.

Then we define specifics of the above described process by choosing S, Δ_{12} and Δ_{21} (w.l.o.g.) and apply it to C. The resulting Nash stable component C' with $\mathscr{T}^{C'}$ featuring $r^{C'} = r^C$ and therefore also the same self-investment as C is depicted on the right hand side.

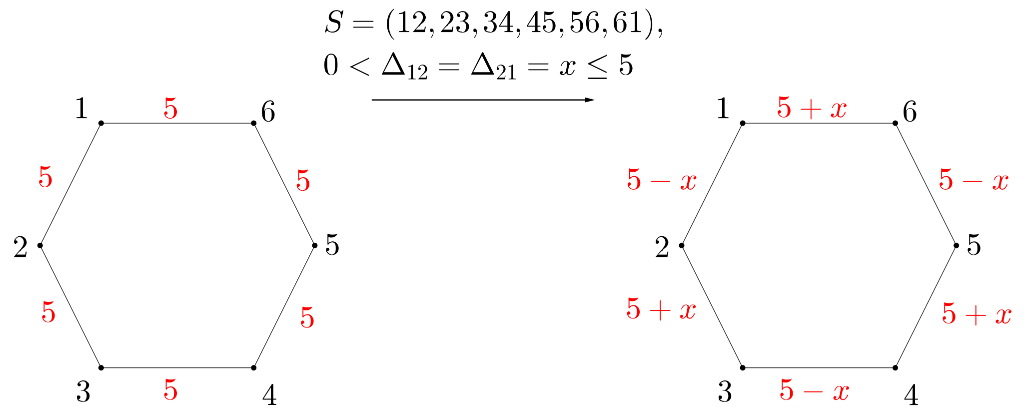

Figure 2: $T = 24$, $r^C = 1$, $t^*_{ii} = 14$ for $i \in N^C$.

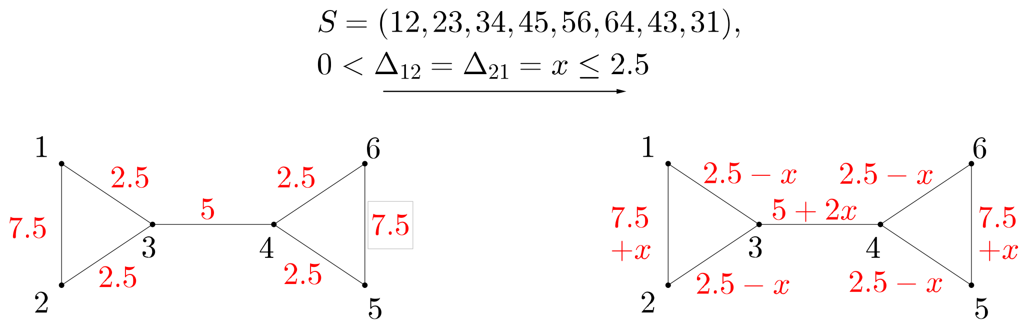

Figure 3: $T = 24$, $r^C = 1$, $t^*_{ii} = 14$ for $i \in N^C$.

The examples highlight the social interdependence in friendship networks. After changing the intensity of one friendship, friendship intensities towards other people and also friendships between people who were not part of the initial change have to be adjusted in order to reach a new Nash stable feasible allocation of time.

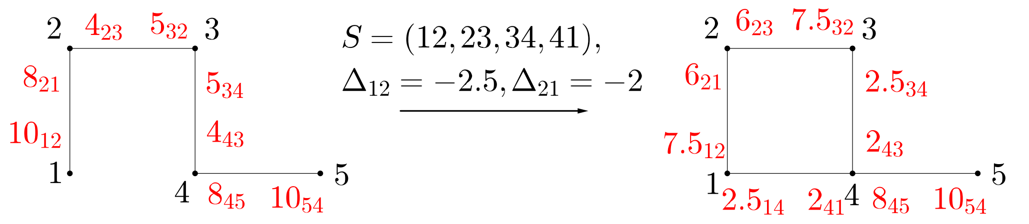

Figure 4: $T = 24$, $r^C = \frac{5}{4}$, $t^*_{ll} = 14$ for $l \in N^C$ and $t^*_{kk} = 12$ for $k \in N^C$.

4 Pairwise stable networks

As we have seen in the previous section there exists a large finite number of network structures in terms of link existence for which we can find a Nash stable distribution of time and there exist infinitely many Nash stable distributions of time. The network consists of a finite set of nodes N with which we can construct finitely many different component structures with a Nash stable distribution of time. Nash stable distributions of time can be varied continuously on many component structures and are hence infinitely many (c.f. Figure 2 and 3). Applying the stricter concept of strong pairwise stability after Bloch and Dutta (2009) reduces the number of stable networks significantly.

Definition 2. *A network $\mathscr{T}$ is strongly pairwise stable if it is Nash stable and if there are no two individuals (i, j) which would be both strictly better off by a joint deviation from (t_i, t_j) to (t'_i, t'_j).*

Lemma 4.1. *No Nash stable reciprocal component with $n^C \geq 3$ is strongly pairwise stable.*

A formal proof is provided in the appendix. The intuition behind Lemma 4.1 is that in a reciprocal network, given every other agent's investment strategy, two agents always have the strict incentive to team up and spend more time which they take from another friendship with each other. This way the two deviating agents receive more total attention than before. A unilateral reduction of investment in a reciprocal relationship decreases one agent's utility from friendship less than a bilateral raise of reciprocal investment increases one agent's utility.

Lemma 4.2. *No Nash stable non-reciprocal component is strongly pairwise stable.*

A formal proof is provided in the appendix. The intuition behind this result is that in every non-reciprocal Nash stable component two type l agents - and there are at least two type l agents - have a strict incentive to establish a reciprocal relationship by reducing their investment into a friend k and spending this time with each other.

From the two previous lemmata a more general statement about the pairwise stability of the whole network can be derived.

Proposition 4.3. *No network containing a component with* $n^C \geq 3$ *is strongly pairwise stable.*

Since a Nash stable network consists of Nash stable components it cannot be pairwise stable if its components are not pairwise stable.

We could also derive further statements about networks only containing components of size $n^C \leq 2$. Yet, this requires to look at different special cases and does not yield any insights beyond the main result of this section: Two agents always have a strict incentive to jointly establish or invest more into a reciprocal link between each other if each one can take the necessary time from another existing reciprocal or non-reciprocal link to a type k agent. Thereby these two agents receive a higher overall time investment.

5 Agent utility and welfare

In this section we will first compare the utility between the different types of agents in Nash stable components. In Nash stable non-reciprocal component every agent is either type l or type k, and in a Nash stable reciprocal component every agent is type i.

In a Nash stable non-reciprocal component the utility of agent k , u_k^*, is

$$u_k^* = \sum_{\substack{l \in L^C, \\ t_{kl}^*, t_{lk}^* > 0}} a t_{kl}^{*\beta} t_{lk}^{*1-\beta} + f(t_{kk}^*)$$

and the utility of agent l, u_l^*, is

$$u_l^* = \sum_{\substack{k \in K^C, \\ t_{lk}^*, t_{kl}^* > 0}} a t_{lk}^{*\beta} t_{kl}^{*1-\beta} + f(t_{ll}^*)$$

where $t_{kk}^* < t_{ll}^*$ and $\frac{t_{lk}^*}{t_{kl}^*} > 1$ for each friendship kl.

In Nash stable reciprocal component, the utility of each agent i, u_i^*, is

$$u_i^* = \sum_{\substack{j \neq i, \\ t_{ij}^*, t_{ji}^* > 0}} a t_{ij}^{*\beta} t_{ji}^{*1-\beta} + f(t_{ii}^*)$$

with $\frac{t_{ji}^*}{t_{ij}^*} = 1$ for all friendships ij. We know that $t_{kk}^* < t_{ii}^* < t_{ll}^*$.

Proposition 5.1. $u_k^* > u_i^* > u_l^*$.

We can show that a type k agent who spends most time on socializing and receives most attention ($TS_k^* > TS_i^* > TS_l^*$ and $\frac{t_{lk}^*}{t_{kl}^*} > 1$ for all of k's friendships, i.e. the amount of social time received by k is even higher than TS_k^*) has the highest utility in a Nash stable component. A type i agent who spends the second highest amount of time on socializing and receives the same amount of social time as she invest ($t_{ij}^* = t_{ji}^*$ for all of i's friendships) has the second highest utility. A type l agent who invests least time into socializing and receives even less time than she invests ($\frac{t_{lk}^*}{t_{kl}^*} > 1$ for all of l's friendships) has the lowest level of utility. A formal proof for Proposition 5.1 can be found in the appendix.

Whether a reciprocal or non-reciprocal Nash stable solution for a component of given size n^C dominates in terms of welfare is not obvious. The welfare from a non-reciprocal Nash stable solution ($r > 1$) is

$$W_{r>1} = \left|K^C\right| u_k^* + \left|L^C\right| u_l^*,$$

and from reciprocal Nash stable solution ($r = 1$)

$$W_{r=1} = \left|N^C\right| u_i^* = \left(\left|K^C\right| + \left|L^C\right|\right) u_i^*.$$

Since $u_k^* > u_i^* > u_l^*$ and $\left|K^C\right| < \left|L^C\right|$, we cannot determine if $W_{r>1} > W_{r=1}$ or $W_{r>1} < W_{r=1}$ without further information. Yet, we can show that none of the Nash stable equilibria is welfare maximal.

We will call a network which is welfare maximal an "efficient" network. Herein, we follow Jackson and Wolinsky (1996) who term a network as "strongly efficient" if and only if it maximizes the sum of individual utilities. We use the same notion but will just refer to it as "efficient" (as does Jackson (2008)) and not as "strongly efficient". The sum of individual utilities in network $\mathscr{T}$, $W(\mathscr{T})$, is

$$W(\mathscr{T}) = \sum_{i \in N} \left(\sum_{j \neq i} a t_{ij}^{\beta} t_{ji}^{1-\beta} + f(t_{ii}) \right).$$

The efficient network is characterized by the solution to the following optimization problem:

$$\max_{t_{11}, t_{12}, \ldots, t_{nn}} W(\mathscr{T})$$
$$s.t. \quad \sum_j t_{ij} = T \quad \text{for all } i.$$

The time constraint has to be satisfied for each agent individually since time endowment is agent-specific and we do not grant the social planner the power to transfer time. We will denote values of the efficient network with superscript W. Equally to individual best responses in the Nash equilibrium of the model, the social planner chooses $t_{ij}^W = 0$ if $t_{ji}^W = 0$, $t_{ij}^W > 0$ if $t_{ji}^W > 0$, and $t_{ii}^W > 0$.

If $t_{ij}^W = 0$ and $t_{ji}^W = 0$ the link ij does not exist in $\mathscr{T}^W$; if $t_{ij}^W > 0$ and $t_{ji}^W > 0$ the link ij exists. Hence, the social planner has to decide which links to establish and then implement the efficient level of time investment on existing links. From the Lagrangian and after rearranging we derive the FOCs for the efficient levels of time investment on existing links:

$$a\beta \left(\frac{t_{ji}}{t_{ij}}\right)^{1-\beta} + a(1-\beta)\left(\frac{t_{ji}}{t_{ij}}\right)^{\beta} = f'(t_{ii}) \qquad \text{for all } i \text{ with } j \neq i \tag{3}$$

$$\sum_j t_{ij} = T \qquad \text{for all } i \tag{4}$$

As in the Nash equilibrium case, there are again two types of $\mathscr{T}$ which solve the system of linear equations. These two types of $\mathscr{T}$ are candidates for an efficient structure of time investments.

First, $\mathscr{T}$ with $t_{ij} = t_{ji}$ for all $i \neq j$, t_{ii} such that $a = f'(t_{ii})$, and $\sum_j t_{ij} = T$ for all i is a solution to the FOCs. This is again a reciprocal situation with $r = 1$. Compared to the Nash stable time investment $\mathscr{T}^*$ with $r = 1$, $t_{ii} < t_{ii}^*$ because $a = f'(t_{ii}) > a\beta = f'(t_{ii}^*)$ and f is strictly concave. The intuition behind this is that the agent does not take into account the positive externality of her social time investment on her friends when choosing the level of self-investment. The social planner does and hence chooses a higher amount of social time than the individual agent does.

A second solution to the FOCs is $\mathscr{T}$ which describes a bipartite network of two sets of nodes L and K with the same characteristics as a Nash stable component with $r > 1$, except for a lower self-investment of both types. To be more precise, $\mathscr{T}$ which describes a bipartite network of two sets of nodes L and K with $|L| > |K|$, $\frac{t_{lk}}{t_{kl}} = r > 1$, and $t_{ll} > t_{kk}$ with $f'(t_{ll}) = a\beta\left(\frac{1}{r}\right)^{1-\beta} + a(1-\beta)\left(\frac{1}{r}\right)^{\beta}$ and $f'(t_{kk}) = a\beta r^{1-\beta} + a(1-\beta)r^{\beta}$ is another solution. Note that for a given $r > 1$, $f'(t_{ll}) > f'(t_{ll}^*)$ and $f'(t_{kk}) > f'(t_{kk}^*)$. Hence, $t_{ll} < t_{ll}^*$ and $t_{kk} < t_{kk}^*$. Since this second candidate time distribution implies a bipartition of the network, the non-existence of links between i and j in L and i and j in K must be efficient for this second solution to be an efficient time distribution. Yet, we can show that it strictly improves welfare if we establish a reciprocal link between i and j in L after reducing their self-investment. The reason is that every agent l has a relatively low marginal utility from friendships and self-investment in the candidate time distribution. Because of the concavity of utility in self-investment, reducing self-investment by some $\epsilon > 0$ and establishing a reciprocal link with $j \in L$ which implies a linear increase in utility lead to a net utility gain. Therefore, the second candidate time distribution is not an efficient time distribution. A rigorous proof is given in the proof of the following Proposition 5.2 in the appendix.

On the other hand, the non-existence of links in the reciprocal candidate solution is efficient because of the constant returns to scale to time investment in friendship. Due to the linear dependence of utility from friendship on time investment if $r = 1$, the sum of utilities does not change when we consider different link existence structures which satisfy the FOCs. Hence, all networks of which the time distribution satisfies the reciprocal candidate solution are efficient.

Proposition 5.2. *A network is efficient if and only if* $t_{ij}^W = t_{ji}^W$ *for* $j \neq i$, t_{ii}^W *such that* $f'(t_{ii}^W) = a$, *and* $\sum_j t_{ij}^W = T$ *for all* i.

An efficient reciprocal time distribution exists for any link existence structure for which also a Nash stable time distribution exists. The efficient time distribution has the same characteristics as the Nash stable reciprocal distribution except for a lower level of self-investment which, however, does not affect the possible structures of underlying link existence. The efficient time distribution with $t_{ii}^W = c$ resembles a Nash stable distribution with $r = 1$ and $t_{ii}^* = c$.

6 Conclusion

In this paper we investigated which networks are Nash stable, pairwise stable and efficient when each agent can invest a limited amount of time into herself and into friendships with other agents in the network. An agent's utility from self-investment followed a strictly concave function and utility from friendship a Cobb-Douglas function taking both agents' investments as inputs. Our particular interest lay in how the direct social interdependence and the budget constraint affect the shape of stable and efficient networks.

We found two different types of Nash stable components: a reciprocal one with symmetric agents and a non-reciprocal one with a smaller group of agents investing less into themselves and less into each friendship than their friend, and a larger group of agents investing more into themselves and more into a friendship than their friend. Disturbing a Nash stable network revealed the consequences of social interdependence: after changing the intensity of one friendship, other friendships between agents not involved in the initial change need to be adjusted in order to reach a new Nash stable situation. No component with $n^C \geq 3$ is pairwise stable because there always exist two agents who have a strict incentive to establish a reciprocal link by reducing their investment into some other friend. We can interpret this as an incentive to free ride on a friend's time investment and to be disloyal in order to receive a larger total amount of time. The efficient network is reciprocal but features a higher self-investment than the Nash stable reciprocal one because positive externalities which arise through the social interdependence are taken into account.

The following points are among the limitations of this paper and constitute interesting questions for further investigation. An obvious and promising avenue for future research is to pursue the analysis with heterogeneous agents. The models lends itself easily to the extension of introducing heterogeneity in agents' intrinsic values, the αs, in the marginal utility from self-investment, the f's, and the productivity of an agent's own investment relative to the other agent's investment, the βs. This would allow to derive results on assortativity for example.

Moreover, it would be interesting to analyze if the results presented in this paper still hold if we abandon the assumption of constant returns to scale to joint investment in friendship, and rather allow for a more general value production in friendship of the form $a_j t_{ij}^{\alpha} t_{ji}^{\beta}$, with $0 < \alpha < 1$ and $0 < \beta < 1$. So agents would still face decreasing marginal returns to individual investment. Yet, we would account for decreasing, constant and increasing returns to scale of joint investment.

Appendix

Proof of Lemma 3.3. From equation 1, if $t_{ji} > 0$ and $t_{ki} > 0$, in Nash equilibrium it is true that $a\beta \left(\frac{t_{ji}}{t_{ij}}\right)^{1-\beta} = a\beta \left(\frac{t_{ki}}{t_{ik}}\right)^{1-\beta} = f'(t_{ii})$ from the FOCs of agent i. This is only true if $\frac{t_{ji}}{t_{ij}} = \frac{t_{ki}}{t_{ik}}$. Hence, $\frac{t_{ji}}{t_{ij}} = \frac{t_{ki}}{t_{ik}} := r$. Agent j solves an optimization problem equivalent agent i's. Thus, if $t_{ij} > 0$ and $t_{lj} > 0$, we know from the FOCs of agent j that in Nash equilibrium it must be true that $a\beta \left(\frac{t_{ij}}{t_{ji}}\right)^{1-\beta} = a\beta \left(\frac{t_{lj}}{t_{jl}}\right)^{1-\beta} = f'(t_{jj}) \Rightarrow \frac{t_{ij}}{t_{ji}} = \frac{t_{lj}}{t_{jl}} = \frac{1}{r}$. The same argument applies to agent k. □

Proof of Lemma 3.5. Without loss of generality, let agents $1, 2, ..., k$ constitute a cycle of odd length. In other words, the links $12, 23, ..., k1$ exist and k is an odd numbered agent. By Lemma 3.3, we know that $\frac{t^*_{21}}{t^*_{12}} = \frac{t^*_{k1}}{t^*_{1k}} = r$. Then, for every even numbered agent e, $\frac{t^*_{e-1,e}}{t^*_{e,e-1}} = \frac{t^*_{e+1,e}}{t^*_{e,e+1}} = \frac{1}{r}$, and for every odd numbered agent o, $\frac{t^*_{o-1,o}}{t^*_{o,o-1}} = \frac{t^*_{o+1,o}}{t^*_{o,o+1}} = r$. Hence, for agent k, $\frac{t^*_{k-1,k}}{t^*_{k,k-1}} = \frac{t^*_{1k}}{t^*_{k1}} = r$. Thus, $\frac{t^*_{k1}}{t^*_{1k}} = \frac{t^*_{1k}}{t^*_{k1}} = r$ which is only true if $t^*_{1k} = t^*_{k1}$. This, implies that $r = 1$. Since all agents in one component are connected the ratio of time investments of all friendships of each agent is either equal to r or $\frac{1}{r}$. As $r = 1$ the ratio of time investments in every friendship is equal to 1. Hence, all agents in the component reciprocate each others' time investment, i.e. in Nash equilibrium $t^*_{ij} = t^*_{ji}$ for $i \neq j$. □

Proof of Theorem 3.11. In order to show how Theorem 3.11 derives from Theorem 35.1 by Schrijver (2004, p. 584) we first state the referenced theorem, and introduce necessary definitions. Then we show how it translates into Theorem 3.11. Definitions in Schrijver (2004) necessary for Theorem 35.1 are:

- A *function* $w : Y \rightarrow \mathbb{R}$ is a vector w in $\mathbb{R}^Y$ with components denoted by $w(y)$ or w_y. For any $U \subseteq Y$, $w(U) := \sum_{y \in U} w(y)$.
- $G = (V, E)$ is a graph G with set of vertices V and set of edges E.
- $E[X, Y]$ is the set of edges xy in E with $x \in X$ and $y \in Y$. $E[Y]$ is the set of edges ij in E with $i, j \in Y$.

- $G[T]$ with $T \subseteq V$ is the subgraph induced by T: $G[T] = (T, E[T])$.
- $\delta(v)$ is the set of edges incident with vertex $v \in V$.
- $a, b \in \mathbb{Z}^V$ with $a \leq b$ and $d, c \in \mathbb{Z}^E$ with $d < c$.
- x is a function $x \in \mathbb{Z}^{\mathbb{E}}$ such that (i) $d(e) \leq x_e \leq c(e)$ for all $e \in E$ and (ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for all $v \in V$.

Theorem 35.1, Schrijver (2004, p. 584). Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ with $a \leq b$ and $d, c \in \mathbb{Z}^E$ with $d < c$. Then there exists an $x \in \mathbb{Z}^E$ if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and
(35.2) $b(K) + c(E[K, W]) + d(E[K, U])$
odd is at most
(35.3) $b(U) - 2d(E[U]) - d(E[T, U]) - a(W) + 2c(E[W]) + c(E[T, W])$.

Theorem 35.1 can be applied in the following way to answer our question for which component structures a Nash stable reciprocal distribution of time investments exists:

Our graph G is the component structure G^C with $V = N^C$ and $E = E^C$. In our case, $x_{ij} = st^*_{ij} = st^*_{ji} > 0$ for edge (link) $ij \in E^C$ and $x(\delta((v)) = s\sum_{j \neq i} t^*_{ij} = sTS^*_i$ with s being a scalar to scale our time distribution adequately. Hence, we set $a = b = sTS^*_i$. Moreover, we take s sufficiently large. We will see later what "sufficiently large" means. We set $d = 1$ since we require $x_e > 0$ and take c as the "largest" integer, so symbolically $c = \infty$. Hence, a Nash stable reciprocal time distribution for G^C exists if and only if x exists .

Next, we show how Theorem 35.1 with these specifications reduces to Theorem 3.11 in our case. First, we observe that (35.3) becomes

$$sTS^*_i\,|U| - 2\,|E[U]| - |E[T, U]| - sTS^*_i\,|W| + 2\infty\,|E[W]| + \infty\,|E[T, W]|\,.$$

For every partition T, U, W of N^C with $E[W] \neq \emptyset$ and/or $E[T, W] \neq \emptyset$, the number of components K with $b(K) = a(K)$ and (35.2) odd (in the following we will refer to these as "such components") is always smaller than (35.3) since (35.3) is ∞ and the number of components is finite (definitely smaller than n^C). Hence, it remains to check if for every partition T, U, W with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ the number of such components is smaller than (35.3) which in this case reduces to

$$sTS_i^* \left|U\right| - 2\left|E[U]\right| - \left|E[T, U]\right| - sTS_i^* \left|W\right|.$$

If $|U| < |W|$ then the number of such components should be less than some negative integer which is impossible. Hence, no x exists.

If $|U| = |W|$ and $E[U] \neq \emptyset$ and/or $E[T, U] \neq \emptyset$ again no x exists. If $|U| = |W|$, $E[U] = \emptyset$ and $E[T, U] = \emptyset$, then the number of such components must be equal to zero in order for x to exist. This is indeed fulfilled in our model for every partition T, U, W in which $E[W] = \emptyset$, $E[T, W] = \emptyset$, $|U| = |W|$, $E[U] = \emptyset$ and $E[T, U] = \emptyset$: Since G^C is a component all vertices $v \in N^C$ are connected. If $E[T, W] = \emptyset$ and $T \neq \emptyset$, then there must exist a connection between T and U otherwise $v \in T$ would not be connected to the component and would hence not be part of the component. But $E[T, U] = \emptyset$ and hence $T = \emptyset$. So the number of such components in T is zero. Thus, for every partition T, U, W in which $E[W] = \emptyset$, $E[T, W] = \emptyset$, $|U| = |W|$, $E[U] = \emptyset$ and $E[T, U] = \emptyset$, x exists. This result corresponds to 2. of our Theorem 3.11.

If $|U| > |W|$ then the number of such components is always at most as large as (35.3) since we chose s sufficiently large. Hence, x exists. This result corresponds to 1. of our Theorem 3.11.

So far we have shown that in our case x exists if and only if for every partition T, U, W with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ either 1) $|U| > |W|$, or 2) $|U| = |W|$, $E[U] = \emptyset$ and $E[T, U] = \emptyset$. It remains to be shown that it is sufficient to only look at partitions T, U, W with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ where W is the set of isolates in $G^C - U$. In other words, we need to show that if 1) or 2) are satisfied for every partition T, U, W with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ where W is the set of isolates in $G^C - U$, then 1) or 2) are also satisfied for every partition T, U, W with $E[W] = \emptyset$ and

$E[T,W]=\emptyset$ in which T and W both contain isolates of G^C-U. The following proof is an adaptation of the sufficiency proof for Corollary 35.1a in Schrijver (2004). If T contains a singleton component $K=\{v\}$, then moving $\{v\}$ to W reduces the number of such components by at most 1, while $sTS_i^*\,|U|-2\,|E[U]|-|E[T,U]|-sTS_i^*\,|W|$ decreases by $sTS_i^*-E[\{v\},U]>1$ for s sufficiently large – the constraint (35.3) becomes tighter by moving isolates from T to W. Thus, if 1) or 2) are satisfied for every partition T,U,W with W being the set of all isolates of G^C-U, then 1) or 2) are also satisfied for every partition T,U,W with $E[W]=\emptyset$, $E[T,W]=\emptyset$ and both T and W containing isolates.

Now, we have arrived at our Theorem 3.11:

There exists a Nash stable reciprocal distribution of time investments for $G^C=(N^C,E^C)$ with $t_{ij}^*>0$ if $ij\in E^C$, and $t_{ij}^*=0$ if $ij\notin E^C$ if and only if for every $U\subseteq N^C$ with W being the set of isolates of G^C-U either

1. $|U|>|W|$,

or

2. $|U|=|W|$ and for every link $ij\in E^C$ with $i\in U$ it is true that $j\in W$.

□

Proof of Lemma 4.1. We check if there exists a strict incentive for a pairwise joint deviation of agent i and j in a Nash stable reciprocal component with $n^C\geq 3$. Suppose agent i and j deviate by increasing each of their investments into their friendship by $\epsilon>0$. Agent i (j) takes the time for the joint deviation from her friendship with another agent m (n) ($m\neq n$ or $m=n$). Then only i's utility from her friendship with j and m changes while any other utility is not affected. Then, i has a strict incentive to jointly deviate if

$$at_{ij}^{\beta}t_{ji}^{1-\beta}+at_{im}^{\beta}t_{mi}^{1-\beta}<a\left(t_{ij}+\epsilon\right)^{\beta}\left(t_{ji}+\epsilon\right)^{1-\beta}+a\left(t_{im}-\epsilon\right)^{\beta}t_{mi}^{1-\beta}. \qquad (5)$$

We know that in a Nash stable reciprocal component $t_{ij} = t_{ji}$. Hence, equation 5 becomes

$$t_{ij} + t_{im} < (t_{ij} + \epsilon) + (t_{im} - \epsilon)^{\beta}\, t_{im}^{1-\beta}$$

$$t_{im} - (t_{im} - \epsilon)^{\beta} t_{im}^{1-\beta} < \epsilon.$$

Since $t_{im} - (t_{im} - \epsilon)^{\beta} t_{im}^{1-\beta} < t_{im} - (t_{im} - \epsilon)^{\beta}\,(t_{im} - \epsilon)^{1-\beta} = \epsilon$, agent i has a strict incentive to deviate jointly. Because of symmetry the same is true for agent j. Thus, no reciprocal component with $n^C \geq 3$ is strongly pairwise stable. □

Proof of Lemma 4.2. A Nash stable non-reciprocal component is bipartite with two sets of agents L^C and K^C with $\left|L^C\right| > \left|K^C\right|$ (thus $\left|L^C\right| \geq 2$), $\frac{t_{lk}^*}{t_{kl}^*} = r > 1$, and $t_{ll}^* > t_{kk}^*$. To prove Lemma 4.2 we show that two agents i and j in L^C have a strict incentive to jointly deviate by establishing a reciprocal link with a mutual time investment $\epsilon > 0$ between each other (due to the bipartiteness no link exists before the deviation between i and j) and by reducing their time investment into the friendship with an agent $k \in K^C$ by this amount. Agent i has a strict incentive to decrease her time investment into k by ϵ and to jointly establish a reciprocal friendship with j if

$$t_{ik}^{\beta} t_{ki}^{1-\beta} < \epsilon^{\beta} \epsilon^{1-\beta} + (t_{ik} - \epsilon)^{\beta} t_{ki}^{1-\beta}$$

$$\Leftrightarrow \left(t_{ik}^{\beta} - (t_{ik} - \epsilon)^{\beta}\right) t_{ki}^{1-\beta} < \epsilon.$$

As $t_{ik} > t_{ki}$, it is sufficient to show that

$$\left(t_{ik}^{\beta} - (t_{ik} - \epsilon)^{\beta}\right) t_{ik}^{1-\beta} \leq \epsilon.$$

We have already shown that is true in the proof of Lemma 4.1. The same holds true for agent j since i and j are symmetric. □

Proof of Proposition 5.1. We will first show that $u_k^* > u_i^*$, second that $u_i^* > u_l^*$ and third conclude that $u_k^* > u_l^*$. Let us rewrite u_k^* first:

$$
\begin{aligned}
u_k^* &= \sum_{\substack{l \in L^C, \\ t_{kl}^*, t_{lk}^* > 0}} a t_{kl}^{*\beta} t_{lk}^{*1-\beta} + f(t_{kk}^*) \\
&= \sum_{\substack{l \in L^C, \\ t_{kl}^*, t_{lk}^* > 0}} a t_{kl}^* \frac{t_{kl}^{*\beta}}{t_{kl}^*} t_{lk}^{*1-\beta} + f(t_{kk}^*) \\
&= a \sum_{\substack{l \in L^C, \\ t_{kl}^*, t_{lk}^* > 0}} t_{kl}^* \left(\frac{t_{lk}^*}{t_{kl}^*} \right)^{1-\beta} + f(t_{kk}^*) \\
&= aTS_k^* r^{1-\beta} + f(t_{kk}^*)
\end{aligned}
$$

The last equality follows from the property that in a Nash stable reciprocal component, $\frac{t_{lk}^*}{t_{kl}^*} = r > 1$ for all existing links kl. Similarly, we can rewrite u_i^* as $u_i^* = aTS_i^* + f(t_{ii}^*)$ by making use of the fact that $\frac{t_{ji}^*}{t_{ij}^*} = 1$ in a Nash stable reciprocal component for all existing links ij. Now observe that

$$
\begin{aligned}
aTS_k^* r^{1-\beta} + f(t_{kk}^*) &> aTS_i^* + f(t_{ii}^*) \\
\Leftrightarrow a \left(TS_k^* r^{1-\beta} - TS_i^* \right) &> f(t_{ii}^*) - f(t_{kk}^*) = \int_{t_{kk}^*}^{t_{ii}^*} f'(t) dt.
\end{aligned}
$$

Since f is strictly concave and $t_{ii}^* > t_{kk}^*$, it is sufficient to show that

$$
a \left(TS_k^* r^{1-\beta} - TS_i^* \right) > f'(t_{kk}^*)(t_{ii}^* - t_{kk}^*).
$$

From the FOCs we know that $f'(t_{kk}^*) = a\beta r^{1-\beta}$ and hence

$$
a \left(TS_k^* r^{1-\beta} - TS_i^* \right) > a\beta r^{1-\beta} (TS_k^* - TS_i^*)
$$

which is true. Thus, $u_k^* > u_i^*$.

With the same argument we used above adjusted to the specific case, we can rewrite u_l^* as $u_l^* = aTS_l^* \left(\frac{1}{r}\right)^{1-\beta} + f(t_{ll}^*)$ and then show that $u_i^* > u_l^*$.

Proof of Proposition 5.2. We only show in this proof that a bipartition given the $r > 1$ solution is not efficient because two agents i and j in L can both be made strictly better off by reducing their self-investment by ϵ and establishing a reciprocal link with a mutual investment of ϵ between them. In every $r > 1$ solution there are at least two of these agents because $|L| \geq 2$. The change in utility due to this operation for agent i is

$$\Delta u_i = a\epsilon + f(t_{ll} - \epsilon) - f(t_{ll}) = a\epsilon - \int_{t_{ll}-\epsilon}^{t_{ll}} f'(x)dx.$$

Because of the strict concavity of f

$$a\epsilon - \int_{t_{ll}-\epsilon}^{t_{ll}} f'(x)dx > a\epsilon - f'(t_{ll} - \epsilon)\epsilon$$

if $\epsilon > 0$.

Moreover, $f'(t_{ll}) = a\beta\left(\frac{t_{kl}}{t_{lk}}\right)^{1-\beta} + a(1-\beta)\left(\frac{t_{kl}}{t_{lk}}\right)^{\beta} < a$ as $\frac{t_{kl}}{t_{lk}} < 1$. Hence, for every $\epsilon \in (0, t_{ll} - c]$ with $f'(c) = a$

$$\Delta u_i > a\epsilon - f'(t_{ll} - \epsilon)\epsilon \geq a\epsilon - a\epsilon = 0.$$

Due to symmetry of agent i and j, $\Delta u_j > 0$, too. No other agent besides i and j is affected in her level of utility. Thus, the non-reciprocal solution to the FOCs is not efficient.

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