Multilateral Bargaining in Networks: On the Prevalence of Inefficiencies

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Abstract

We introduce a noncooperative multilateral bargaining model for a network-restricted environment, in which players can communicate only with their neighbors. Each player strategically chooses the bargaining partners among the neighbors to buy out their communication links with upfront transfers. The main theorem characterizes a condition on network structures for efficient equilibria and shows the prevalence of strategic delays. If the underlying network is either *complete* or *circular*, then an efficient stationary subgame perfect equilibrium exists for *all* discount factors: all the players always try to reach an agreement as soon as practicable and hence no strategic delay occurs. In *any other* network, however, an efficient equilibrium is impossible for sufficiently high discount factors because some players strategically delay an agreement. We also provide an example of a *Braess-like paradox*, in which the more links are available, the less links are actually used. Thus, network improvements may decrease social welfare.

keywords: noncooperative bargaining, coalition formation, communication restriction, buyout, network, Braess's Paradox

JEL Classification: C72, C78; D72, D74, D85

1 Introduction

Communication restrictions are imposed to an environment where generating a surplus requires an agreement among three or more players. Consider a simplest possible example. Each of three players has a different but indispensable resource, for instance, labor, land, or capital, to produce a unit surplus. For an informational reason, two 'leaf' players cannot directly communicate with each other, while the 'central' player is connected to all the other players. In such environments, the following natural questions arise: How the surplus might be allocated among the players, taking their stragegic interactions into

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account? How they choose the bargaining partner among their neighbors? Does the more communication links guarantee higher welfere level?

To analyze these questions, we introduce a noncooperative bargaining model in which each player can communicate only with the directly connected players in a given network.[1] In each period, a proposer is randomly selected and the proposer makes an offer specifying a coalition among the neighbors and monetary transfers to each member in the proposed coalition. If all the members in the coalition accept the offer, then the coalition forms and the proposer controls the coalition thereafter, inheriting other members' network connections (See Figure 1). Otherwise, the offer dissolves. The game repeats until the grand-coalition forms, after which the player who controls the grand-coalition obtains the unit surplus. All the players have a common discount factor.[2]

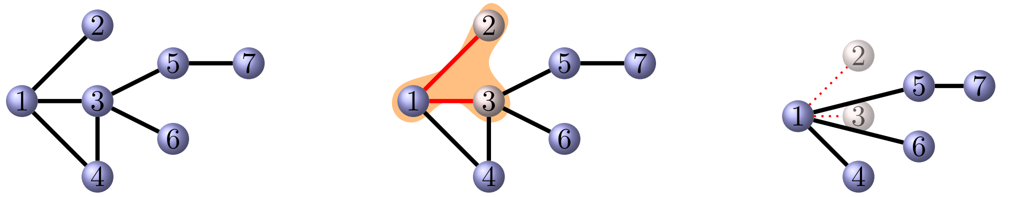
The main result characterizes a condition on network structures for efficient equilibria. If the underlying network is either *complete* or *circular*, then for *any* discount factor there exists an efficient stationary subgame perfect equilibrium. In such an efficient equilibrium, all the players always try to reach an agreement as soon as practicable and hence no strategic delay occurs. In any other network, however, an efficient stationary subgame perfect equilibrium is impossible for sufficiently high discount factors – strategic delay must occur at least some positive probability. For instance in the earlier three-player chain example, in any stationary subgame perfect equilibrium, the central player demands too much and leaf players decline to make an offer even though they are selected as a proposer; and the agreement will be delayed until the central player becomes a proposer. We show that such a strategic delay is prevalent so that at least one player must be better off by strategically delaying an agreement unless the underlying network is either complete or circular.

We also provide an interesting example in which adding a new communication link decreases social welfare. This observation is reminiscent of the *Braess's paradox* which first appeared in Braess (1968).[3] In the original context, the Braess's paradox refers a situation that constructing a new route reduces overall performance when players choose their route selfishly. Analogously in our model, each player strategically chooses commu-

[1] Since Aumann and Dreze (1974), cooperation restrictions have been studied mainly in cooperative games. Myerson (1977) uses a network to described the structure of cooperation restrictions.

[2] This model can be extended to a more general class of network-restricted games. In this paper, however, in order to concentrate on network structures and control network-irrelevant factors, we confine our attention to a unanimity game but allow any possible network. Lee (2014) considers general transferable utility environments but not network restrictions.

[3] See Braess et al. (2005) for translation from the original German. In recent years, numerous studies on Braess's paradox have been made particularly in computer science and related disciplines. See Roughgarden (2005) for more details.



(a) An initial network (b) Player 1 forms $\{1, 2, 3\}$ (c) The induced network

Figure 1: A Coalition Formation in a Network

nication links to use for bargaining. Similarly, the more links are available, the less links are actually used. As the result, network improvements decrease social welfare.[4] To our best knowledge, this is the first observation of an analog of the Braess's paradox in the bargaining literature.

The model has two important features which distinguish it from the existing noncooperative bargaining models in networks. First, we allow *strategic coalition formation* so that each player can choose the partners to bargain with. In the literature, however, players' strategic interaction is limited in a randomly selected meeting. A bilateral meeting (Manea, 2011a,b; Abreu and Manea, 2012a,b) or a multilateral meeting (Nguyen, 2012) randomly occurs, then the players in the random meeting bargain over their joint surplus.[5] As Hart and Mas-Colell (1996) pointed out, however, a random-meeting-model does not entirely capture players' strategic behaviors and strategic decision on coalition formation should also be considered.

Next, we allow players to *buy out* other players and it enables them to gradually form a coalition.[6] In the Manea/Abreu-Manea/Nguyen model, given a coalition, players' strategic decision is limited on how to split the coalitional surplus: All the players in the coalition, once they reach an agreement, must exit the game and they are excluded in further bargaining. Therefore, those models are not applicable to an environment in which gradual coalition formation is inevitable to generate a surplus.[7] On the other hand, when players can buy out other players as an intermediate step, they not only consider the surplus of the current coalition itself, but also take into account the subsequent bargaining

[4] In the Braess's paradox with a traffic network context, all the players are worse off; while in this bargaining game in a communication network, some players may be better off even though overall performance deteriorates.

[5] Abreu and Manea (2012a) also consider an alternative model in which a proposer chooses a bargaining partner. However, their model is still limited to a bilateral bargaining.

[6] The notion of *buyout option* is based on Gul (1989). In this sense, we succeed Gul (1989). On the other hand, since his model assumes random meeting, we develop the model by allowing strategic coalition formation.

[7] For instance, in a four-player circle, since no one can immediately form the grand-coalition, the surplus cannot be realized in their models.

game. Thus players may even form a zero-surplus coalition strategically.

The paper is organized as follows. In section 2, we introduce a noncooperative multilateral bargaining model for a network-restricted environment. Section 3 provides the main characterization result with leading examples. Section 4 considers an alternative model in which players cannot trade their chances of being a proposer. Missing proofs are presented in Appendix.

2 A Model

2.1 Networks

A *network* (or a graph) $g = (N, E)$ consists of a finite set $N = \{1, 2, \cdots, n\}$ of *player*s (or nodes) and a set E of *link*s (or edges) of N. When $g = (N, E)$ is not the only network under consideration, the notations $N(g)$ and $E(g)$ are occasionally used for the player set and the link set rather than N and E to emphasize the underlying network g. Through this paper, we assume that g is simple[8] and connected. Given $g = (N, E)$ and $S \subseteq N$, a *subgraph* restricted on S is $g_{|S} = (S, \{ij \in E \mid \{i, j\} \subseteq S\})$. The (closed) *neighborhood* of $i \in N$ is given by $N_i(g) \equiv \{j \in N \mid \exists ij \in E\} \cup \{i\}$. Let $\deg_i(g) \equiv |N_i(g)| - 1$ be a *degree* of i and $d(i, j; g)$ be a (geodesic) *distance* between i and j in g.

A set $S \subseteq N$ is *dominating* in g if, for all $i \in N$, either $i \in S$ or there exists $j \in S$ such that $ij \in E$. A player $i \in N$ is *dominating* in g if $\{i\}$ is a dominating set. Let $D(g)$ be a set of dominating players in g. A dominating set S is *minimal* if no proper subset is a dominating set. A network is *trivial* if $|N(g)| = 1$. For any integer $k = 2, \cdots, n-1$, a network is *k-regular* if $\deg_i(g) = k$ for all $i \in N(g)$. A network g is *complete* if it is $(n-1)$-regular, or equivalently if $D(g) = N(g)$. A connected network g is *circular* if it is 2-regular.[9]

A *complete cover* of g is a collection $\mathcal{M}$ of subsets of $N(g)$, such that, $\cup\mathcal{M} = N(g)$ and $g_{|M}$ is a complete network for all $M \in \mathcal{M}$. A *complete covering number* of g is the minimum cardinality of a complete cover of g. A *minimal complete cover* is a complete cover of which cardinality is minimum.

2.2 A Noncooperative Bargaining Game

A *noncooperative bargaining game*, or shortly a *game*, is a triple $\Gamma = (g, p, \delta)$, where g is a underlying network, $p \in \mathbb{R}^{|N|}_{++}$ is an initial *recognition probability* with $\sum_{i \in N} p_i = 1$, and

[8]A simple network is an unweighted and undirected network without loops or multiple edges.

[9]A *circular network* (or a *circle*) should not be confused with a *cycle in a network*. A circular network is a network that consists of a single cycle.

$0 < \delta < 1$ is a common discount factor.

A game $\Gamma = (g, p, \delta)$ proceeds as follows. In each period, Nature selects a player $i \in N$ as a proposer with probability p_i. Then, the proposer i makes an offer, that is, i strategically chooses a pair (S, y) of a coalition $S \subseteq N_i(g)$ and monetary transfers $y \in \mathbb{R}_+^{|N|}$ with $\sum_{j \in N} y_j = 0$. Each respondent $j \in S \setminus \{i\}$ sequentially either accepts the offer or rejects it.[10] If any $j \in S \setminus \{i\}$ rejects the offer, then the offer dissolves and all the players repeat the same game in the next period. If each $j \in S \setminus \{i\}$ accepts the offer, then i *buys out* $S \setminus \{i\}$, that is, each respondent $j \in S \setminus \{i\}$ leaves the game with receiving y_j from the proposer i and the remaining players $(N \setminus S) \cup \{i\}$ play the subsequent game $\Gamma^{(i,S)}$ in the next period. All the players have a common discount factor δ.

After i buys out $S \setminus \{i\}$, or i forms S, the subsequent game $\Gamma^{(i,S)} = \left(g^{(i,S)}, p^{(i,S)}, \delta\right)$ is defined in the following way:

i) The induced network $g^{(i,S)} = \left(N^{(i,S)}, E^{(i,S)}\right)$, where $N^{(i,S)} = (N \setminus S) \cup \{i\}$ and

$$E^{(i,S)} = \{ij \mid (\exists i'j \in E)\ i' \in S \text{ and } j \in N \setminus S\} \bigcup \{jk \mid (\exists jk \in E)\ j, k \in N \setminus S\}.$$

That is, after i's S-formation, $S \setminus \{i\}$ leaves the network, but i inherits all the network connections from S.

ii) The induced recognition probability $p^{(i,S)}$:

$$p_j^{(i,S)} = \begin{cases} p_S & \text{if } j = i \\ p_j & \text{if } j \in N \setminus S \\ 0 & \text{if } j \in S \setminus \{i\}. \end{cases}$$

That is, the proposer i takes the respondents' chances of being a proposer as well.

The game continues until only one player remains, after which the last player acquires one unit of surplus. When the game ends in finite period T, the history h specifies a finite sequence $\tilde{y}(h) = \{y^t(h)\}_{t=0}^T$ of monetary transfers and the last player $i^*(h) \in N$. Given $\Gamma = (g, p, \delta)$ and a history h, player i's discounted sum of expected payoffs is

$$U_i(h) = \sum_{t=0}^{T} \delta^t y_i^t(h) + \delta^T \mathbb{1}(i = i^*(h)).$$

If the game does not end within finite periods, then the history h induces a sequence $\tilde{y}(h)$ of monetary transfers without determining the last player, and hence player i's discounted sum of expected payoffs is

$$U_i(h) = \sum_{t=0}^{\infty} \delta^t y_i^t(h).$$

[10]The result does not depend on the order of responses.

2.3 Coalitional States

A (coalitional) *state* π is a partition of N, specifying a set of *active players* $N^\pi \subseteq N$. For each active player $i \in N^\pi$, i's partition block $[i]_\pi$ represents the players i together with players whom he has previously bought out. Denote π° by the initial state, that is, $N^{\pi^\circ} = N$ and $[i]_{\pi^\circ} = \{i\}$ for all $i \in N$. A state π is *terminal* if $|N^\pi| = 1$.

A state π is *feasible* in g, if there exists a sequence of coalition formations $\{(i_\ell, S_\ell)\}_{\ell=1}^L$ such that $i_1 \in N$ and $S_{i_1} \subseteq N_{i_1}$; and $i_\ell \in N^{(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}$ and $S_\ell \subseteq N_{i_\ell}^{(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}$ for all $\ell = 2, \cdots, L$; and $N^\pi = N^{(i_1,S_1)\cdots(i_L,S_L)}$. Let $\Pi(g)$ be a set of all feasible states in g. For each $\pi \in \Pi(g)$, the induced network $g^\pi = (N^\pi, E^\pi)$ is uniquely determined by

$$E^\pi \equiv \bigcup_{i \in N^\pi} \Big\{ ij \mid \exists i'j' \in E \ (i' \in [i]_\pi \text{ and } j' \in [j]_\pi) \Big\},$$

and the induced recognition probability p^π is determined by

$$p_i^\pi = \begin{cases} \sum_{j \in [i]_\pi} p_j & \text{if } i \in N^\pi \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

When there is no danger of confusion, we omit π° in notations, for instance, $g^{\pi^\circ} = g$, $g^{\pi^\circ(i,S)} = g^{(i,S)}$, and so on. The description of the underlying network g may also be omitted, when it is clear. For notational simplicity, for any $v \in \mathbb{R}^{|N|}$ and any $S \subseteq N$, we denote $v_S = \sum_{j \in S} v_j$.

2.4 Stationary Subgame Perfect Equilibria

We focus on stationary subgame perfect equilibria. A stationary strategy depends only on the current coalitional state and within-period histories, but not the histories of past periods. The existence of a stationary subgame perfect equilibrium is known in the literature including Eraslan (2002) and Eraslan and McLennan (2013). See Lee (2014) for the formal description of stationary strategies. In the literature, instead of considering all the possible stationary strategies, a simple stationary strategy, namely a *cutoff strategy*, is usually accepted.

A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ consists of a *value profile* $\mathbf{x} = \{\{x_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$ and a *coalition formation strategy profile* $\mathbf{q} = \{\{q_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$, where $x_i^\pi \in \mathbb{R}$ and $q_i^\pi \in \Delta(2^{N_i^\pi})$ for each $\pi \in \Pi(g)$.[11] A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ specifies the behaviors of an active player $i \in N^\pi$: in the following way:

[11]Through this paper, for a finite set X, $\Delta(X)$ is the set of all possible probability measures in X.

- player i proposes (S, y) with probability $q_i^\pi(S)$ such that

$$y_k = \begin{cases} \delta x_k^\pi & \text{if } k \in S \setminus \{i\} \\ -\delta x_{S\setminus\{i\}}^\pi & \text{if } k = i \\ 0 & \text{otherwise;} \end{cases}$$

- player i accepts any offer (S, y) with $i \in S$ if and only if $y_i \geq \delta x_i^\pi$.

Note that player i can decline to make an offer by choosing $S = \{i\}$. A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ induces a probability measure $\mu_{\mathbf{x},\mathbf{q}}$ on the set of all possible histories. Given history h, let $\tilde{\pi}(h) = \{\pi^t(h)\}_{t=0}^T$ be a sequence of states which is determined by h. Given $(\mathbf{x}, \mathbf{q})$, define the set of *inducible states*:

$$\Pi_{\mathbf{x},\mathbf{q}}(g) = \{\pi \in \Pi(g) \mid (\exists h \ \exists t) \ \mu_{\mathbf{x},\mathbf{q}}(h) > 0 \text{ and } \pi = \pi^t(h)\}.$$

Given $\mathbf{x}$, for each $\pi \in \Pi(g)$, $i \in N^\pi$, and $S \subseteq N_i^\pi$, define a player i's *excess surplus* of S-formation:

$$e_i^\pi(S, \mathbf{x}) = \begin{cases} \delta x_i^{\pi(i,S)} - \delta x_S^\pi & \text{if } S \subsetneq N^\pi \\ 1 - \delta x_{N^\pi}^\pi & \text{if } S = N^\pi. \end{cases}$$

Let $\mathcal{D}_i^\pi(\mathbf{x}) = \operatorname{argmax}_{S\subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a *demand set* of player i in π and $m_i^\pi(\mathbf{x}) = \max_{S\subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a (net) *proposal gain* of player i in π. Given a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$, define an active player i's *continuation payoff* in π:

$$\begin{aligned} u_i^\pi(\mathbf{x}, \mathbf{q}) &= p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) e_i^\pi(S, \mathbf{x}) + \sum_{j\in N^\pi} p_j^\pi \left(\sum_{S:i\in S\subseteq N^\pi} q_j^\pi(S)\delta x_i^\pi + \delta \left(\sum_{S:i\notin S\subseteq N^\pi} q_j^\pi(S) x_i^{\pi(j,S)} \right) \right) \\ &= p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) e_i^\pi(S, \mathbf{x}) + \delta \left(\sum_{j\in N^\pi} p_j^\pi \sum_{S\subseteq N^\pi} q_j^\pi(S) \left(\mathbb{1}(i \in S) x_i^\pi + \mathbb{1}(i \notin S) x_i^{\pi(j,S)} \right) \right). \end{aligned} \tag{2}$$

We close this section with two important lemmas which provide fundamental tools for our analysis. Lemma 1 shows that any stationary subgame perfect equilibrium can be uniquely represented by a cutoff strategy equilibrium in terms of a equilibrium payoff vector. Thus, when we are interested in players' equilibrium payoffs or efficiency, without loss of generality, we may consider only cutoff strategy equilibria. Through this paper, an equilibrium refers a cutoff strategy equilibrium. Lemma 2 characterizes a cutoff strategy equilibrium with two tractable conditions, *optimality* and *consistency*. More general versions of the proofs can be found in Lee (2014).

Lemma 1. *For any stationary subgame perfect equilibrium, there exists a cutoff strategy equilibrium which yields the same equilibrium payoff vector.*

Lemma 2. *A cutoff strategy profile* $(\mathbf{x}, \mathbf{q})$ *is an stationary subgame perfect equilibrium if and only if, for all* $\pi \in \Pi$ *and* $i \in N^\pi$*, the following two conditions hold,*

i) **Optimality:** $q_i^\pi \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$*; and*

ii) **Consistency:** $x_i^\pi = u_i^\pi(\mathbf{x}, \mathbf{q})$.

3 Efficient Equilibria

In this section, we characterize a necessary and sufficient condition on network structures for efficient equilibria. Given g, define a *maximum coalition formation strategy profile* $\bar{\mathbf{q}} = \{\{\bar{q}_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi(g)}$ with

$$\bar{q}_i^\pi(S) = \begin{cases} 1 & \text{if } S = N_i^\pi \\ 0 & \text{otherwise,} \end{cases}$$

that is, for each state $\pi \in \Pi(g)$, each proposer $i \in N^\pi$ chooses a maximum coalition N_i^π to bargain with. Given $\Gamma = (g, p, \delta)$, let $\bar{u}(\Gamma)$ be a maximum welfare. Note that $\bar{u}(\Gamma)$ is obtained by any cutoff strategy profile involves with a maximum coalition formation strategy profile. A strategy profile $(\mathbf{x}, \mathbf{q})$ is *efficient* if

$$\sum_{i \in N} u_i(\mathbf{x}, \mathbf{q}) = \bar{u}(\Gamma). \tag{3}$$

Example 1. Let $N = \{1, 2, 3, 4\}$. Consider two game $\Gamma = (g, p, \delta)$ and $\Gamma' = (g', p, \delta)$, where $g = (N, \{12, 23, 34, 41\})$ is a circular network and $g' = (N, \{12, 23, 34, 41, 13\})$ is a chordal network. It is easy to see $\bar{u}(\Gamma) = \delta$ and $\bar{u}(\Gamma') = (p_1 + p_3) + \delta(p_2 + p_4)$.

An efficient strategy profile does not necessarily consist of maximum coalition formation strategies. For each $\pi \in \Pi(g)$, define a set of i's coalitions which maximizes the sum of players' expected payoffs in the subsequent state:

$$\mathcal{E}_i^\pi \equiv \operatorname*{argmax}_{S \subseteq N_i^\pi} \bar{u}(\Gamma^{\pi(i,S)}).$$

Lemma 3. *Given* $\Gamma = (g, p, \delta)$*, an equilibrium* $(\mathbf{x}, \mathbf{q})$ *is efficient if and only if,*

$$\forall \pi \in \Pi_{\mathbf{x},\mathbf{q}}(g) \quad \forall i \in N^\pi \quad q_i^\pi \in \Delta(\mathcal{E}_i^\pi).$$

Proof. If $|N(g)| = 2$, then the statement is obviously true. As an induction hypothesis, suppose the statement is true for any less-than-n-player game. Consider g with $|N(g)| = n$. For any $\pi \in \Pi(g)$, observe that summing (2) over N^π yields

$$\begin{aligned} \sum_{i \in N^\pi} u_i^\pi(\mathbf{x}, \mathbf{q}) &= \sum_{i \in N^\pi} p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) \left[e_i^\pi(S, \mathbf{x}) + \delta \left(\sum_{j \in S} x_j^\pi + \sum_{j \notin S} x_j^{\pi(i,S)} \right) \right] \\ &= \sum_{i \in N^\pi} p_i^\pi \sum_{S \subseteq N^\pi} q_i^\pi(S) X^{\pi(i,S)}, \end{aligned} \tag{4}$$

where $X^{\pi(i,N^\pi)} = 1$ and $X^{\pi(i,S)} = \delta \sum_{j\in N^{\pi(i,S)}} x_j^{\pi(i,S)}$ for all $S \subsetneq N^\pi$.

Sufficiency: Let $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. By the consistency condition, for all $S \subsetneq N^\pi$,

$$\sum_{j\in N^{\pi(i,S)}} x_j^{\pi(i,S)} = \sum_{j\in N^{\pi(i,S)}} u_j^{\pi(i,S)}(\mathbf{x}, \mathbf{q}).$$

Since $(\mathbf{x}, \mathbf{q})$ is efficient, the induction hypothesis and the definition of efficiency yield $X^{\pi(i,S)} = \delta\bar{u}(\Gamma^{\pi(i,S)})$ for all $S \subsetneq N^\pi$. Suppose for contradiction that there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$, $i \in N^\pi$, and $S, S' \subseteq N_i^\pi$ such that $q_i^\pi(S) > 0$ and $\bar{u}(\Gamma^{\pi(i,S)}) < \bar{u}(\Gamma^{\pi(i,S')})$. Then i can strictly improve the sum of the players' payoff by putting more weight on S' in his coalition formation strategy and hence q_i^π cannot be a part of an efficient equilibrium.

Necessity: Given g, $\pi \in \Pi(g)$, and $(\mathbf{x}, \mathbf{q})$, define a partial strategy profile $(\mathbf{x}_{|\pi}, \mathbf{q}_{|\pi}) = \{(x^{\pi'}, q^{\pi'})\}_{\pi'\in\Pi(g^\pi)}$. By induction hypothesis, for all $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g) \setminus \{\pi^\circ\}$, $(\mathbf{x}_{|\pi}, \mathbf{q}_{|\pi})$ is an efficient equilibrium for a game with g^π. Consider the initial state. By (4), in order to maximize $\sum_{i\in N} u_i(\mathbf{x}, \mathbf{q})$, each player i must maximize $\sum_{S\subseteq N} q_i(S)X^{(i,S)}$. Since, for all $i \in N$ and all $S \in \mathcal{E}_i$, $(\mathbf{x}_{|(i,S)}, \mathbf{q}_{|(i,S)})$ is an efficient equilibrium for a game with $g^{(i,S)}$, the condition $q_i \in \Delta(\mathcal{E}_i)$ maximizes $\sum_{i\in N} u_i(\mathbf{x}, \mathbf{q})$ and hence $(\mathbf{x}, \mathbf{q})$ is efficient. □

We are ready to state our main theorem, which characterizes a condition on network structures for efficient equilibria.

Theorem 1. *An efficient stationary subgame perfect equilibrium exists for all discount factors if and only if the underlying network is either* complete *or* circular*.*

We prove the theorem through four propositions. For the sufficient condition, in subsection 3.1, we construct an efficient equilibrium in a complete network (Proposition 1) and in a circular network (Proposition 2). Moreover, in a complete network, the equilibrium payoff vector is unique and hence any stationary subgame perfect equilibrium is efficient. For the necessary condition, Proposition 3 proves the inefficiency result for a specific class of networks, namely *pre-complete networks*, in subsection 3.2. That is, if the underlying network is pre-complete and non-circular, then any stationary subgame perfect equilibrium is inefficient for a sufficiently high discount factor. In subsection 3.3, Proposition 4 completes the necessary condition by showing that, for any game with an incomplete non-circular network, any efficient strategy induces a pre-complete non-circular network with positive probability.

3.1 The Sufficient Condition

First, we consider a complete network. Proposition 1 shows that a unanimous agreement is always immediately reached for any p and δ. Furthermore, any equilibrium is efficient and its payoff vector equals to p. Let $\mathbf{p} = \{\{p_i^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi}$.

Proposition 1. *Let g be a complete network. For any $\Gamma = (g, p, \delta)$,*

i) *there exists a cutoff strategy equilibrium $(\mathbf{p}, \bar{\mathbf{q}})$;*

ii) *for any equilibrium, the equilibrium payoff vector equals to p.*

Example 2 (A Three-Player Complete Network)**.** Let g be a complete network with $N(g) = \{i, j, k\}$ and p be an initial recognition probability. In the first period, a proposer i forms a grand-coalition by buying out other two players at the prices of δp_j and δp_k. Thus the unit surplus belongs to i and his payoff is $1-(p_j+p_k)\delta = 1-(1-p_i)\delta$. Thus, before a proposer is selected, player i's expected payoff is $p_i\cdot(1-(1-p_i)\delta)+(p_j+p_k)\cdot\delta p_i = p_i$. $\square$

Next, in a circular network, we construct an efficient equilibrium in which each player always forms a maximum coalition and the equilibrium payoff vector is proportional to the initial recognition probability. Recall that $\lfloor x \rfloor$ is the largest integer not greater than x.

Proposition 2. *Let g be a circular network. For any $\Gamma = (g, p, \delta)$, there exists a cutoff strategy equilibrium $(\mathbf{x}, \bar{\mathbf{q}})$, where for all $\pi \in \Pi(g)$ and all $i \in N^\pi$,*

$$x_i^\pi = \delta^{\left\lfloor \frac{|N^\pi|}{2} \right\rfloor - 1} p_i^\pi. \tag{5}$$

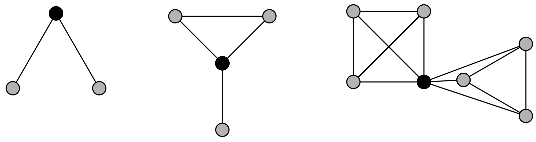
Example 3 (A Four-Player Circular Network)**.** Let g be a circular network with $|N(g)| = 4$. For all π with $2 \leq |N^\pi| \leq 3$, since g^π is complete, the equilibrium strategies in a non-initial state π are $x^\pi = p^\pi$ and $q^\pi = \bar{q}^\pi$, which are consistent with (5). For the initial state, take any $i \in N$ and let $N_i = \{i, j, k\}$. For any $\{i\} \subsetneq S \subseteq N_i$, since S-formation induces a complete network, the excess surplus from S-formation is $e_i(S, \mathbf{x}) = p_S\delta - \delta x_S = \delta(1-\delta)p_S$, which implies $\mathcal{D}_i = \{N_i\}$. For all $\ell \in N$, then $q_\ell(N_\ell) = 1$. Thus, we have $\sum_{\ell\in N} p_\ell \sum_{S\ni i} q_\ell(S) = p_{N_i}$ and $\sum_{\ell\in N} p_\ell \sum_{S\not\ni i} q_\ell(S) = 1 - p_{N_i}$. Therefore, i's expected payoff is:

$$u_i(\mathbf{x}, \bar{\mathbf{q}}) = p_i \cdot \delta(1-\delta)p_{N_i} + \delta\left[p_{N_i} \cdot \delta p_i + (1 - p_{N_i}) \cdot p_i\right] = \delta p_i,$$

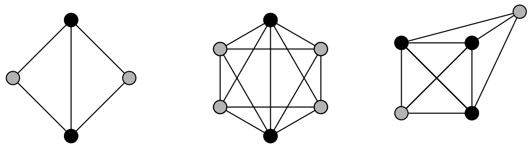
which satisfies consistency condition. $\square$

3.2 The Necessary Condition : Pre-complete Networks

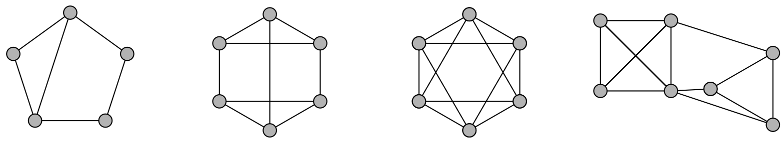
To prove the necessary condition, we will show that any efficient strategy profile cannot be an equilibrium in any incomplete non-circular network if the discount factor is sufficiently high. First, we need to define a special class of networks, namely *pre-complete* networks,



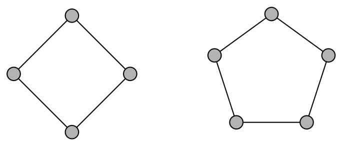
(a) Single Dominating Player: Inefficient



(b) Multiple Dominating Players: Inefficient



(c) No Dominating Player (Non-circular): Inefficient



(d) Circular: Efficient.

Figure 2: Examples of Pre-complete Networks: A dark node represents a dominating player.

in which all the players can induce a complete network. Given g, denote a set of i's feasible coalitions which yield a complete network by

$$\mathcal{C}_i(g) = \{S \subseteq N_i(g) \mid g^{(i,S)} \text{ is complete.}\}$$

Definition 1. A graph g is *pre-complete* if

$$\forall i \in N(g) \quad \{i\} \notin \mathcal{C}_i(g) \neq \emptyset.$$

See Figure 2 for examples of pre-complete networks.

Proposition 3. *Let g be a pre-complete non-circular network. For any p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, any efficient strategy profile $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium in $\Gamma = (g, p, \delta)$.*

In a pre-complete network, there may or may not be exist a dominating player. We divide the proof into two disjoint cases, $D(g) \neq \emptyset$ and $D(g) = \emptyset$.

3.2.1 Case 1: Dominating Players

We provide two leading examples to illustrate an occurrence of a strategic delay. Based on the examples, we discuss a *Braess-Like paradox*. Then, we prove Proposition 3 in a case of $D(g) \neq \emptyset$.

The first example is of a three-player chain, in which there is only one dominating player. In such a chain, the unique dominating player has a stronger bargaining power than the other players so that her value is too high for the other players to buy her out. Thus, when non-dominating players are recognized as a proposer, they decline to make an offer and a delay occurs.

Example 4 (A Chain)**.** Let $g = (\{1,2,3\},\{12,13\})$. First, we show an impossibility of an efficient equilibrium. Suppose there exists an efficient equilibrium $(\mathbf{x},\mathbf{q})$. Then player 1 is always included in a proposed coalition, that is, $q_1(N) = q_2(\{1,2\}) = q_3(\{1,3\}) = 1$. Thus player 1's expected payoff is $u_1(\mathbf{x},\mathbf{q}) = p_1(1-\delta x_N) + \delta x_1$. Since $x_1 = u_1(\mathbf{x},\mathbf{q})$ and $x_N = p_1 + (1-p_1)\delta$, it follows $(1-\delta)x_1 = p_1(1-\delta(p_1+(1-p_1)\delta))$, or equivalently,

$$x_1 = p_1(1+(1-p_1)\delta). \tag{6}$$

On the other hand, player 2's expected payoff is

$$u_2(\mathbf{x},\mathbf{q}) = p_2 m_2(\mathbf{x}) + \delta((p_1+p_2)x_2 + p_3p_2) \geq \delta(1-p_3)x_2 + p_3p_2\delta.$$

By consistency, we have $x_2 \geq \frac{\delta p_2 p_3}{1-\delta(1-p_3)}$ and similarly $x_3 \geq \frac{\delta p_2 p_3}{1-\delta(1-p_2)}$. Together with (6), it requires that

$$x_N \geq p_1(1+(1-p_1)\delta) + \frac{\delta p_2 p_3}{1-\delta(1-p_3)} + \frac{\delta p_2 p_3}{1-\delta(1-p_2)}.$$

To see a contradiction, as δ converges to 1, observe that the right-hand side converges to $1 + p_1(1-p_1)$, which is strictly greater than 1 as long as $p_1 > 0$. However, x_N never exceeds 1. Thus, for a sufficiently high δ, the efficient strategy profile $(\mathbf{x},\mathbf{q})$ cannot be an equilibrium.

Next, we construct an inefficient equilibrium. Let $\bar{\delta} = \max\left\{\frac{p_2}{(p_1+p_2)(1-p_1)}, \frac{p_3}{(p_1+p_3)(1-p_1)}\right\}$ so that $\bar{\delta} < 1$. Consider a strategy profile $(\mathbf{x},\mathbf{q})$ such that

- $x_1 = \frac{p_1}{1-(1-p_1)\delta}$; $x_2 = x_3 = 0$; and
- $q_1(N) = q_2(\{2\}) = q_3(\{3\}) = 1$,

and in any two-player subgame the active players follow the strategy according to Proposition 1. Since player 2 and player 3 decline to be a proposer in the initial state, the strategy profile is inefficient. To see that $(\mathbf{x},\mathbf{q})$ constructs an equilibrium for $\delta > \bar{\delta}$, due to Lemma 2, it suffices to verify the following two conditions.

i) *Optimality:* Calculate each player's excess surpluses. It is easy to see that $e_1(N, \mathbf{x}) > 0$ and $e_i(\{i\}, \mathbf{x}) = 0$ for all $i \in N$. For all $i \in \{1, 2\}$, due to Proposition 1, $x_i^{(i,\{1,2\})} = p_1 + p_2$, and hence

$$\begin{aligned} e_i(\{1,2\}, \mathbf{x}) &= \delta(p_1 + p_2) - \delta(x_1 + x_2) = \delta(p_1 + p_2) - \delta\left(\tfrac{p_1}{1-(1-p_1)\delta} + 0\right) \\ &= \tfrac{\delta}{1-(1-p_1)\delta}\left(p_2 - (p_1 + p_2)(1 - p_1)\delta\right). \end{aligned}$$

Then, $\delta > \bar{\delta}$ implies $e_i(\{1,2\}, \mathbf{x}) < 0$. Similarly, we have $e_i(\{1,3\}, \mathbf{x}) < 0$ for all $i \in \{1, 3\}$. Given $\mathbf{x}$, therefore, $\mathcal{D}_1 = \{N\}$, $\mathcal{D}_2 = \{\{2\}\}$, and $\mathcal{D}_3 = \{\{3\}\}$.

ii) *Consistency:* Compute each player's expected payoff:

- $u_1(\mathbf{x}, \mathbf{q}) = p_1 e(N, \mathbf{x}) + \delta x_1 = p_1(1 - \delta x_1) + \delta x_1 = p_1 + (1 - p_1)\delta\left(\tfrac{p_1}{1-(1-p_1)\delta}\right) = \tfrac{p_1}{1-(1-p_1)\delta}$
- $u_2(\mathbf{x}, \mathbf{q}) = p_2 e(\{2\}, \mathbf{x}) + \delta x_2 = p_2 \cdot 0 + \delta \cdot 0 = 0$
- $u_3(\mathbf{x}, \mathbf{q}) = p_3 e(\{3\}, \mathbf{x}) + \delta x_3 = p_3 \cdot 0 + \delta \cdot 0 = 0$.

Therefore, $u_i(\mathbf{x}, \mathbf{q}) = x_i$ for all $i \in N$. □

Even if there are multiple dominating players, as see (b) in Figure 2, they can generate an additional advantage by forming a coalition with other dominating players and splitting non-dominating players into two isolated groups. In the next example, we construct an equilibrium in a chordal network in which there are two dominating players.

Example 5 (A Chordal Network)**.** Let $g = (\{1, 2, 3, 4\}, \{12, 23, 34, 41, 13\})$ and $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Suppose $\delta > \bar{\delta} \approx 0.91$.[12] We construct an equilibrium $(\mathbf{x}, \mathbf{q})$ such that

- $x_1 = x_3 = \frac{(6-\delta)\delta}{4(4-\delta)(2-\delta)}$; $x_2 = x_4 = \frac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)}$;
- $q_1(\{1, 3\}) = q_3(\{1, 3\}) = 1$; $q_2(\{1, 2\}) = q_2(\{2, 3\}) = q_4(\{1, 4\}) = q_4(\{3, 4\}) = \frac{1}{2}$.

In any subgame in which the number of active players is less than or equal to three, they follows the equilibrium strategies according to Proposition 1 and Example 4. Note that the equilibrium welfare is $x_N = \frac{\delta(3-\delta)}{2(2-\delta)}$. The equilibrium payoff vector converges to $\left(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12}\right)$ as $\delta \to 1$. Now we verify the equilibrium conditions.

i) *Odd Players' Optimality:* Since $\delta > \frac{3}{4}$, Example 4 implies that $x_1^{(1,\{1,3\})} = \frac{p_1+p_3}{1-(1-p_1-p_3)\delta} = \frac{1}{2-\delta}$ and $x_1^{(1,\{1,3\})} = x_4^{(1,\{1,3\})} = 0$. Given $\mathbf{x}$, calculate player 1's excess surpluses:

- $e_1(\{1, 2\}, \mathbf{x}) = \delta x_1^{(1,\{1,2\})} - \delta(x_1 + x_2) = \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)}$

[12] Note that $\bar{\delta}$ is a solution to $\delta(8 - 8\delta + \delta^2) = (4 - \delta)(1 - \delta)(4 + 2\delta - \delta^2)$.

- $e_1(\{1,3\},\mathbf{x}) = \delta x_1^{(1,\{1,3\})} - \delta(x_1+x_3) = \frac{\delta(8-8\delta+\delta^2)}{2(2-\delta)(4-\delta)}$
- $e_1(\{1,2,4\},\mathbf{x}) = \delta x_1^{(1,\{1,2,4\})} - \delta(x_1+x_2+x_4) = \frac{\delta(6-6\delta+\delta^2)}{2(4-\delta)}$
- $e_1(N,\mathbf{x}) = 1-\delta x_N = \frac{(1-\delta)(4+2\delta-\delta^2)}{2(2-\delta)}$

Given $e_1(S,\mathbf{x})$ for all $S \subseteq N_1$, it is routine to see that $\delta > \bar{\delta}$ implies $\mathcal{D}_1(\mathbf{x}) = \{\{1,3\}\}$. Similarly, we also have $\mathcal{D}_3(\mathbf{x}) = \{\{1,3\}\}$.

ii) *Even Players' Optimality:* For any $\{2\} \subsetneq S \subseteq N_2$, player 2's S-formation induces a complete network. Thus, given $\mathbf{x}$, one can compute player 2's excess surpluses:

- $e_2(\{1,2\},\mathbf{x}) = e_2(\{2,3\},\mathbf{x}) = \delta x_2^{(2,\{1,2\})} - \delta(x_1+x_2) = \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)}$
- $e(\{1,2,3\},\mathbf{x}) = \delta x_2^{(2,\{1,2,3\})} - \delta(x_1+x_2+x_3) = \frac{\delta(24-36\delta+11\delta^2-\delta^3)}{4(2-\delta)(4-\delta)}$

Observe that $e_2(\{1,2\},\mathbf{x}) = e_2(\{2,3\},\mathbf{x}) > 0$ for all δ; while $e(\{1,2,3\},\mathbf{x})$ is strictly negative if $\delta > \bar{\delta}$. Thus, for any $\delta > \bar{\delta}$, we have $\mathcal{D}_2(\mathbf{x}) = \{\{1,2\},\{2,3\}\}$ and similarly $\mathcal{D}_4(\mathbf{x}) = \{\{1,4\},\{3,4\}\}$.

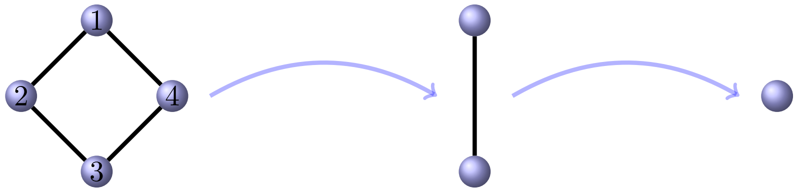
iii) *Consistency:* Given $(\mathbf{x},\mathbf{q})$, compute each players' expected payoffs:

- $$\begin{aligned} u_1(\mathbf{x},\mathbf{q}) &= p_1 e(\{1,3\},\mathbf{x}) + \delta\left[\left(p_1+p_3+\tfrac{1}{2}(p_2+p_4)\right)x_1 + \tfrac{1}{2}(p_2+p_4)p_1\right], \\ &= \tfrac{1}{4}\cdot\tfrac{\delta(8-8\delta+\delta^2)}{2(2-\delta)(4-\delta)} + \delta\left[\tfrac{3}{4}\cdot\tfrac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} + \tfrac{1}{2}\cdot\tfrac{1}{8}\right] \\ &= \tfrac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} = x_1 \end{aligned}$$
- $$\begin{aligned} u_2(\mathbf{x},\mathbf{q}) &= p_2 e(\{1,2\},\mathbf{x}) + \delta\left[p_2x_2 + p_4p_2 + (p_1+p_3)\cdot 0\right], \\ &= \tfrac{1}{4}\cdot\tfrac{\delta(1-\delta)(4-\delta)}{4(2-\delta)} + \delta\left[\tfrac{1}{4}\cdot\tfrac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} + \tfrac{1}{4}\cdot\tfrac{1}{4}\right] \\ &= \tfrac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} = x_2 \end{aligned}$$

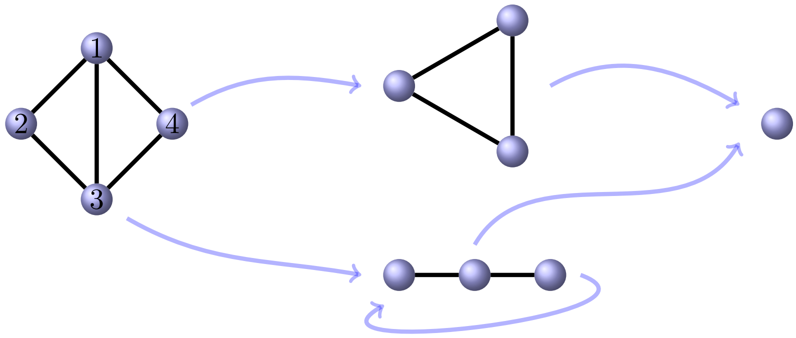
and similarly $u_3(\mathbf{x},\mathbf{q}) = x_3$ and $u_4(\mathbf{x},\mathbf{q}) = x_4$, and hence consistency holds. □

Remark (Braess-Like Paradox). Comparing between Example 3 and Example 5, we observe a negative welfare effect of adding a new communication link.[13] In the four-player circle with $p_i = \frac{1}{4}$ for all $i \in N$, the maximum welfare δ is achieved in an equilibrium. If we add a link between player 1 and player 3 in the circular network, then it becomes a chordal network as in Example 5. Since odd players can form a grand-coalition immediately, the maximum welfare is $\frac{1}{2}(1+\delta)$, which is strictly greater than that of the circular network. However, the equilibrium welfare in Example 5 is $\frac{\delta(3-\delta)}{2(2-\delta)}$ which is strictly less than δ, which is the maximum welfare in the circle. This observation is reminiscent of the

[13]This question has been raised by Vijay Krishna.



(a) Bargaining in a Circular Network (See Example 3): It takes exactly 2 periods for a grand-coalition in any equilibrium. In the first period, any proposer forms a three-player coalition by buying out two neighbors. Then the induced game is of two players.



(b) Bargaining in a Chordal Network (See Example 5): The expected periods for a grand-coalition is strictly greater than 2. In the first period, if the even players are selected as a proposer, then they choose one of the odd players as a bargaining partner to induce a three-player circle. In the circle, grand-coalition immediately forms. However, if the odd players are initially selected as a proposer, then they induce a three-player chain. In the chain, the leaf players decline to make an offer and hence an additional delay occurs with positive probability.

Figure 3: A Braess-Like Paradox

Braess's paradox. In fact, this result does not depend on the initial recognition probability p, as long as $p_2 > 0$ and $p_4 > 0$.

One can observe the negative welfare effect of adding a new link by computing the expected periods for a unanimous agreement. See Figure 3. In the circle, it takes exactly 2 periods for a grand-coalition in the equilibrium. Note that all the players fully use their communication links whenever they are recognized as a proposer. In the chordal network, however, the expected periods for a unanimous agreement is 2.5.[14] If the even players are recognized as a proposer in the first period, then they chooses one of the odd players as a bargaining partner to induce a three-player circle. In the circle, grand-coalition immediately forms. However, if the odd players are initially recognized as a proposer, they induces a three-player chain. In the chain, then the leaf players decline to make an

[14] $$(p_2+p_4)\times 2 + (p_1+p_3)\Big[(p_1+p_3)\times 2 + (p_2+p_4)\Big((p_1+p_3)\times 3 + (p_2+p_4)\big((p_1+p_3)\times 4 + \cdots\big)\Big)\Big]$$
$$= \tfrac{1}{2}\times 2 + \tfrac{1}{4}\times 2 + \tfrac{1}{8}\times 3 + \tfrac{1}{16}\times 4 + \cdots$$
$$= 1 + \sum_{k=2}^{\infty} k\frac{1}{2^k} = 2.5.$$

offer and hence an additional delay occurs with positive probability.

Remark. In the random-proposer bargaining model, the equilibrium may not be unique even in the class of stationary subgame perfect equilibria.[15] However, the equilibrium constructed in Example 3, Example 4, and Example 5 is unique in the class of *symmetric* cutoff-strategy equilibria, in which identical players in terms of a position in a network and a recognition probability play the identical cutoff strategy.

Now, we are ready to prove Proposition 3 in a case of $D(g) \neq \emptyset$. Since g is a pre-complete, note that there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$. Let $J_1(g) = N_{j_1}(g) \setminus D(g)$, $J_2(g) = N_{j_2}(g) \setminus D(g)$, and $J(g) = J_1(g) \cup J_2(g)$. Lemma 4 provides a lower bound of the unique dominating player's expected payoff.

Lemma 4. *Let g be a pre-complete network with $D(g) = \{i\}$. If $(\mathbf{x}, \mathbf{q})$ is an equilibrium of $\Gamma = (g, p, \delta)$, then*

$$x_i \geq p_i + p_i(1 - p_i)\delta. \tag{7}$$

Proof. Step 1: Consider a three-person chain, that is, $J_1 = \{j_1\}$ and $J_2 = \{j_2\}$. Since $x_i^{(j_1, J_1)} = x_i^{(j_2, J_2)} = x_i$ and $u_N(\mathbf{x}, \mathbf{q}) \leq \bar{u}(\Gamma) = p_i + \delta(1 - p_i)$, player i's expected payoff is

$$\begin{aligned} x_i &\geq p_i e_i(N, \mathbf{x}) + \sum_{k \in N} p_k \sum_{S \ni i} q_k(S) \delta x_i + \delta \sum_{k \in N} p_k \sum_{S \not\ni i} q_k(S) x_i \\ &\geq p_i(1 - \delta(p_i + \delta(1 - p_i))) + \delta x_i. \end{aligned}$$

Rearranging the terms, we have the desired result.

Step 2: As an induction hypothesis, assume that for any pre-complete network g' with $D(g') = \{i\}$, $\leq |J_1(g')| \leq a$, and $1 \leq |J_2(g')| \leq b$, $x'_i \geq p'_i + p'_i(1 - p'_i)\delta$. Now we consider a pre-complete network g with $D(g) = \{i\}$, $|J_1(g)| = a$, and $|J_1(g)| = b + 1$. Player i's expected payoff is

$$x_i \geq p_i e_i(N, \mathbf{x}) + \sum_{k \in N} p_k \sum_{S \ni i} q_k(S) \delta x_i + \delta \sum_{k \in N} p_k \sum_{S \not\ni i} q_k(S) x_i^{(k,S)}. \tag{8}$$

For any $k \in N$ and $S \subseteq N$ such that $i \notin S$, the induction hypothesis implies $x_i^{(k,S)} \geq p_i + p_i(1 - p_i)\delta$. Suppose by way of contradiction that $p_i + p_i(1 - p_i)\delta > x_i$. Then, (8) can be written as $x_i \geq p_i\,(1 - \delta(p_i + \delta(1 - p_i))) + \delta x_i$, or equivalently, $x_i > p_i + p_i(1 - p_i)\delta$, which yields a contradiction. Similarly, induction argument completes the proof. $\square$

[15] To overcome multiplicity of equilibria, the uniqueness of equilibrium payoffs has been studied in the random-proposer bargaining model. Eraslan (2002) shows the equilibrium payoff uniqueness for a weighted majority game and Eraslan and McLennan (2013) generalizes this result to a general simple game using fixed point index theorem. Unfortunately, those results cannot be applied to the model in which a player has a buyout option, because a player can expect some partial payoff by forming an intermediate subcoalition and hence the actual characteristic function that the players play is not of a simple game. The uniqueness of stationary equilibrium payoffs is conjectured in a broader class of characteristic function form games, but it still remains as an open question. See Eraslan and McLennan (2013) for a discussion.

Proof of Proposition 3 (Case 1: $D(g) \neq \emptyset$)

Take any $j \in J_1$. Since $(\mathbf{x}, \mathbf{q})$ is efficient, we have $(\forall j' \in J_2)$ $\sum_{S \in \mathcal{C}_{j'}} q_{j'}(S) = 1$ and $(\forall i \in D \cup J_1)$ $\sum_{j \in S \subseteq N} q_i(S) = 1$. Thus, player j's payoff is

$$\begin{aligned} u_j(\mathbf{x}, \mathbf{q}) &= p_j m_j(\mathbf{x}) + \delta(p_D + p_{J_1}) x_j + \delta \sum_{j' \in J_2} p_{j'} \sum_{S \subseteq N} q_{j'}(S) x_j^{(j',S)} \\ &\geq \delta(p_D + p_{J_1}) x_j + \delta p_{J_2} p_j, \end{aligned}$$

which implies that $x_j \geq \frac{p_j p_{J_2} \delta}{1-(1-p_{J_2})\delta}$. Summing j over J_1, we have $x_{J_1} \geq \frac{p_{J_1} p_{J_2} \delta}{1-(1-p_{J_2})\delta}$. Similarly for J_2, we have $x_{J_2} \geq \frac{p_{J_1} p_{J_2} \delta}{1-(1-p_{J_1})\delta}$, and hence

$$x_J = x_{J_1} + x_{J_2} \geq p_{J_1} p_{J_2} \delta \left(\frac{1}{1-(1-p_{J_1})\delta} + \frac{1}{1-(1-p_{J_2})\delta} \right) \tag{9}$$

Now take any $i \in D$. Player i's optimality implies $e_i(N, \mathbf{x}) \geq e_i(D, \mathbf{x})$, or equivalently, $1 - \delta x_N \geq \delta x_i^{(i,D)} - \delta x_D$. Since $g^{(i,D)}$ has a single dominating player, Lemma 4 implies $x_i^{(i,D)} \geq p_D + p_D(1-p_D)\delta$ and it follows that

$$1 - p_D \delta(1 + \delta - p_D \delta) \;\geq\; \delta x_J \tag{10}$$

By (9) and (10), we have

$$1 - p_D \delta(1 + \delta - p_D \delta) \;\geq\; p_{J_1} p_{J_2} \delta^2 \left(\frac{1}{1-(1-p_{J_1})\delta} + \frac{1}{1-(1-p_{J_2})\delta} \right). \tag{11}$$

As $\delta \to 1$, the right hand side of (11) converges to p_J; while the left hand side converges to p_J^2. Since $p_J < 1$, there exists $\bar{\delta} < 1$ such that the inequality (11) yields a contradiction for $\delta > \bar{\delta}$. $\square$

3.2.2 Case 2: No Dominating Player

Now we consider a network without a dominating player. See (c) in Figure 2 for instance. Even in a case of that there is no dominating player, we will show that some players can be a dominating player in the induced network by buying out only a part of their neighbors. Before proving this, some lemmas are presented. First, whenever there is a dominating player in an incomplete network, Lemma 5, Lemma 6, and Lemma 7 show dominating players have some additional bargaining power compared to other non-dominating players. In a network without a dominating player, Lemma 8 shows that each player's payoff should be strictly less than her recognition probability under any efficient equilibrium. For any non-circular pre-complete network without a dominating player, Lemma 9 finds a player who can be a dominating player avoiding a complete network. Combining those lemmas, therefore, when an efficient equilibrium is assumed, at least one player can be strictly better off by strategically delaying a unanimous agreement, which is a contradiction in turn.

Lemma 5. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ) with $\delta < 1$. If $i \in D(g)$ then $x_i > p_i$.*

Proof. If $|N(g)| = 3$, due to Lemma 4, then $x_i \geq p_i + p_i(1-p_i)\delta > p_i$ for any $i \in D$. As an induction hypothesis, suppose the statement is true for any g' with $|N(g')| < n$. Now consider g with $|N(g)| = n$. Take any $i \in D(g)$. For any $k \in N$ and any S such that $i \notin S$, if $g^{(k,S)}$ is complete then $x_i^{(k,S)} = p_i$; and if $g^{(k,S)}$ is incomplete then $x_i^{(k,S)} > p_i$ by the induction hypothesis. Thus, letting $Q_i = \sum_{k \in N} p_k \left(\sum_{S \ni i} q_i(S) + q_k(\{k\}) \right)$, we have $x_i \geq p_i(1 - \delta x_N) + Q_i \delta x_i + \delta(1 - Q_i)p_i$, and hence $x_i \geq p_i + \frac{\delta(1-\delta)}{1-\delta Q_i} p_i > p_i$. □

Lemma 5 says that for any dominating player, her expected payoff is strictly greater than her recognition probability. However, we need a stronger result: the difference between the expected payoff and the recognition probability is strictly positive even in the limit of that the discount factor converges to one. Lemma 6 shows that there exists such a dominating player and Lemma 7 proves it for all dominating players. For notational convenience, denote $\Delta_i = x_i - p_i$ and $\Delta_i^{(j,S)} = x_i^{(j,S)} - p_i^{(j,S)}$. If $g^{(i,S)}$ is complete and non-trivial, by Proposition 1, note that $e_i(S, \mathbf{x}) = \delta(x_i^{(i,S)} - x_S) = \delta(p_S - x_S) = -\delta \Delta_S$.

Lemma 6. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ). There exists $h \in D(g)$ such that*

$$x_h - p_h \geq \frac{p_h \left(p_D(1-p_D)\delta^2 - (1-\delta) \right)}{1 + (|D|-1)p_h \delta}.$$

Furthermore, $\lim_{\delta \to 1}(x_h - p_h) \geq \frac{p_h p_D(1-p_D)}{1+(|D|-1)p_h} > 0$

Proof. Take any $h \in \operatorname{argmax}_{i \in N} \Delta_i$ and let $Q_h = \sum_{i \in N} \sum_{S \ni h} p_i q_i(S)$. For any $i \in N$ and $S \subseteq N$ such that $h \notin S$, since $h \in D(g^{(i,S)})$, Lemma 5 implies $x_h^{(i,S)} \geq p_h$, and hence we have

$$\begin{aligned} x_h &\geq p_h e_h(D, \mathbf{x}) + Q_h \delta x_h + \delta(1 - Q_h)p_h \\ &\geq p_h p_D(1-p_D)\delta^2 - p_h \Delta_D \delta + Q_h \delta(p_h + \Delta_h) + (1-Q_h)\delta p_h \\ &\geq p_h p_D(1-p_D)\delta^2 - p_h |D| \Delta_h \delta + \delta p_h + p_h \Delta_h \delta, \end{aligned} \tag{12}$$

where the second inequality is due to Lemma 4, which implies

$$e_h(D, \mathbf{x}) = \delta \left(x_h^{(h,D)} - x_D \right) \geq \delta(p_D + p_D(1-p_D)\delta - x_D) = p_D(1-p_D)\delta^2 - \Delta_D \delta,$$

and the last inequality comes from $p_h \leq Q_h$, $p_h \geq x_h$, and $\Delta_D \leq |D|\Delta_h$. Subtracting p_h from both sides of (12), we have $\Delta_h \geq \frac{p_h \left(p_D(1-p_D)\delta^2 - (1-\delta) \right)}{1+(|D|-1)p_h \delta}$, as desired. Since $D \subsetneq N$, it must be $p_D < 1$ and hence $\lim_{\delta \to 1} \Delta_h \geq \frac{p_h p_D(1-p_D)}{1+(|D|-1)p_h} > 0$. □

Lemma 7. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ). For any $i \in D(g)$, there exists $\underline{\Delta}_i > 0$ such that $x_i - p_i \geq \underline{\Delta}_i$ as δ converges to 1.*

Proof. We will show that $\lim_{\delta\to 1} \min_{i\in D} \Delta_i > 0$. Let $L = \operatorname{argmin}_{i\in D} \Delta_i$. Since g is a pre-complete, as before there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$, and let $J_1(g) = N_{j_1}(g) \setminus D(g)$, $J_2(g) = N_{j_2}(g) \setminus D(g)$, and $J(g) = J_1(g) \cup J_2(g)$. Recall Lemma 5, which implies $(\forall i \in D)\ \Delta_i > 0$. Thus, for any $j \in J_1$ and $S \subsetneq N$, if $q_j(S) > 0$ then either $S \subseteq J_1$ or $S \cap D = \{\ell\}$ for some $\ell \in L$.

Case 1: Suppose $|J_1| = |J_2| = 1$. Then, for each $j \in J$, $q_j(\{j\}) + \sum_{\ell\in L} q_j(\{j, \ell\}) = 1$, and hence there exists $\ell \in L$ such that $\sum_{j\in J} p_j\,(q_j(\{j\}) + q_j(\{j, \ell\})) \geq \frac{p_J}{|L|}$. Let $Q_\ell = \sum_{j\in J} p_j\,(q_j(\{j\}) + q_j(\{j, \ell\})) + \sum_{i\in D}\sum_{S\ni \ell} p_i q_i(S)$, then $Q_\ell \geq \frac{p_J}{|L|} + p_\ell$. Since $x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) + Q_\ell \delta x_\ell + (1 - Q_\ell)\delta p_\ell$, it follows

$$\Delta_\ell \geq \delta p_\ell(\Delta_\ell + \Delta_J) + \delta\left(\frac{p_J}{|L|} + p_\ell\right)\Delta_\ell - (1-\delta)p_\ell,$$

which implies $\Delta_\ell \geq \frac{-\delta p_\ell \Delta_J - (1-\delta)p_\ell}{1 - \delta\frac{p_J}{|L|}}$. Since $x_N - p_N = \Delta_N < 0$, we have $-\Delta_J \geq \Delta_D \geq \Delta_h$. Thus, by Lemma 6, we have the desired result,

$$\lim_{\delta\to 1} \Delta_\ell \geq -\frac{|L|p_\ell}{|L| - p_J}\Delta_J \geq \frac{|L|p_\ell}{|L| - p_J}\Delta_h \geq \frac{p_\ell p_h p_D(1 - p_D)|L|}{(|L| - p_J)(1 + (|D| - 1)p_h)} > 0.$$

Case 2: As an induction hypothesis, for any pre-complete network g' with $\emptyset \subsetneq D(g') \subsetneq N(g')$ and $1 \leq |J_1(g')| \leq a$ and $1 \leq |J_2(g')| \leq b$ and any equilibrium $(\mathbf{x}', \mathbf{q}')$ of (g', p', δ), assume that $\lim_{\delta\to 1} \min_{i\in D(g')}(x'_i - p'_i) > 0$. Now we consider a pre-complete network g with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $|J_1(g)| = a$ and $|J_2(g)| = b + 1$. Due to the induction hypothesis, there exists $\Delta'_\ell > 0$ such that $\Delta'_\ell \geq \lim_{\delta\to 1}(x_\ell^{(j,J')} - p_\ell)$ for all $\alpha \in \{1, 2\}$, $j \in J_\alpha$, and $J' \subseteq J_\alpha$. Then, we have

$$\begin{aligned} x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) &+ \left(p_\ell + \sum_{\alpha\in\{1,2\}}\sum_{j\in J_\alpha} p_j\,(q_j(\{j\}) + q_j(J_\alpha \cup \{\ell\}))\right)\delta(p_\ell + \Delta_\ell) \\ &+ \left(\sum_{\alpha\in\{1,2\}}\sum_{j\in J_\alpha}\sum_{J'\subseteq J_\alpha} p_j q_j(J')\right)\delta(p_\ell + \Delta'_\ell) + p_{D\setminus\{\ell\}}\delta p_\ell. \quad (13) \end{aligned}$$

If $\lim_{\delta\to 1}\Delta_\ell \geq \Delta'_\ell$, then there is nothing to prove. Suppose that $\lim_{\delta\to 1}\Delta_\ell \leq \Delta'_\ell$. As $\delta \to 1$, then (13) yields $x_\ell \geq -p_\ell\delta\Delta_J + \delta p_\ell + (1 - p_D)\delta\Delta_\ell$, or equivalently, $(1 - (1 - p_D)\delta)\Delta_\ell \geq -\delta p_\ell \Delta_J - (1-\delta)p_\ell$. Take any $h \in \operatorname{argmax}_{i\in D}\Delta_i$. Since $-\Delta_J > \Delta_D > \Delta_h$, it follows that

$$(1 - (1 - p_D)\delta)\Delta_\ell > \delta p_\ell \Delta_h - (1-\delta)p_\ell.$$

By Lemma 5, we have the desired result, $\lim_{\delta\to 1}\Delta_\ell \geq \frac{p_\ell p_h(1-p_D)}{1+(|D|-1)p_h} > 0$. $\square$

Lemma 8. *Let g be a pre-complete network with $D(g) = \emptyset$. If $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium of $\Gamma = (g, p, \delta)$, then for all $i \in N$, $x_i = \delta p_i$.*

Proof. Since g is pre-complete and $(\mathbf{x}, \mathbf{q})$ is efficient, for all $j \in N$, $q_j(S) > 0$ implies $g^{(j,S)}$ is complete. Thus, each player i can expect p_i in the next period by rejecting any offer. Suppose player i gets an offer with $y_i < \delta^2 p_i$. By rejecting y_i, i can be strictly better since the stationary strategy profile guarantees δp_i in the next period. Hence, $x_i \geq \delta p_i$ for all $i \in N$. If there exists $i \in N$ such that $x_i > \delta p_i$, then it must be $x_N > \delta p_N = \delta$, which is infeasible. $\square$

Lemma 9. *Let g be a pre-complete non-circular network with $D(g) = \emptyset$. There exist $i, j \in N(g)$ such that $i \in D(g^{(i,\{i,j\})}) \subsetneq N(g^{(i,\{i,j\})})$.*

Proof. Since g is pre-complete non-circular, its complete covering number is 2. Let $\mathcal{M}$ be a minimal complete cover of g. Since $D(g) = \emptyset$, $\mathcal{M}$ must be disjoint. Given $i \in N$, then let $M_i \in \mathcal{M}$ such that $i \in M_i$. Since $D(g) = \emptyset$, for all $k \in N$, there exists at least one $k' \in M_k^c$ such that $kk' \notin E(g)$, that is, it must be $|M_k^c \setminus N_k(g)| \geq 1$. We will show that there exists $i \in N$ and $j \in M_i^c$ such that $i \in D(g^{(i,\{i,j\})}) \subsetneq N(g^{(i,\{i,j\})})$, by constructing such a pair of i and j in the following two cases. First, suppose there exists $k \in N$ such that $|M_k^c \setminus N_k(g)| \geq 2$. Take $i \in M_k^c \setminus N_k(g)$ and $j \in M_i^c$ with $ij \in E(g)$. Take $i' \in M_k^c \setminus N_k(g)$ with $i' \neq i$. Since $g_{|M_i}$ and $g_{|M_i^c}$ are complete, $i \in D(g^{(i,\{i,j\})})$. Since $d(k, i'; g) = d(k, i'; g^{(i,\{i,j\})}) = 2$, $k \notin N(g^{(i,\{i,j\})})$, as desired. Second, suppose, for all $k \in N$, $|M_k^c \setminus N_k(g)| = 1$. Take any $i \in N$ and $j \in M_i^c$ such that $ij \in E(g)$. Take $k \in M_i \setminus \{i\}$ and $k' \in M_i^c$ such that $d(k, k'; g) = 2$. Again we have $i \in D(g^{(i,\{i,j\})})$ and $d(k, k'; g) = d(k, k'; g^{(i,\{i,j\})}) = 2$, as desired. $\square$

Proof of Proposition 3 (Case 2: $D(g) = \emptyset$)

Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Due to Lemma 8, for all $i \in N$ and all $S \in \mathcal{C}_i$,

$$e_i(S, \mathbf{x}) = \delta \left(x_i^{(i,S)} - x_S \right) = \delta \left(p_S - \delta p_S \right) = \delta(1 - \delta) p_S,$$

which converges to 0 as $\delta \to 1$. By Lemma 9, there exists $i, j \in N(g)$ such that $i \in D(g^{(i,\{i,j\})})$ and $\{i, j\} \notin \mathcal{C}_i$. Due to Lemma 7, there exists $\underline{\Delta}_i$ such that $x_i^{(i,\{i,j\})} - p_i^{(i,\{i,j\})} \geq \underline{\Delta}_i$. By Lemma 8, then we have

$$\begin{aligned} e_i(\{i,j\}, \mathbf{x}) &= \delta \left(x_i^{(i,\{i,j\})} - (x_i + x_j) \right) \\ &\geq \delta \left((p_i^{(i,\{i,j\})} + \underline{\Delta}_i - \delta(p_i + p_j) \right) \\ &= \delta \underline{\Delta}_i + \delta(1 - \delta)(p_i + p_j). \end{aligned}$$

As $\delta \to 1$, note that $e_i(\{i,j\}, \mathbf{x}) \geq \underline{\Delta}_i > 0$. Thus for a sufficiently high δ, $e_i(\{i,j\}, \mathbf{x}) > e_i(S, \mathbf{x})$ for all $S \in \mathcal{C}_i$, which contradicts to optimality of player i. □

3.3 The Necessary Condition : Incomplete Networks

We have considered pre-complete non-circular networks. To complete the necessary condition, we have to allow any incomplete non-circular network. Proposition 4 implies that for any game with an incomplete non-circular network, if the players play efficient strategies, then a pre-complete non-circular network must be induced with positive probability.

Proposition 4. *Let g be an incomplete network. For any efficient strategy profile $(\mathbf{x}, \mathbf{q})$, there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$ such that g^π is a pre-complete network. In addition, if g is a non-circular network, then there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$ such that g^π is a pre-complete non-circular network.*

Proof. Suppose that g is neither pre-complete nor complete. Now we construct a sequence of coalition formations which is consistent with $(\mathbf{x}, \mathbf{q})$ and the sequence induces a pre-complete network. Take $i^* \in \text{argmax}_{i \in N(g)} \deg_i(g)$. Let $I(g) = \{i \in N(g) \mid \mathcal{C}_i(g) = \emptyset\}$. Let $g_1 = g$ and take $i_1 \in \text{argmax}_{i \in I(g_1)} d(i, i^*; g_1)$. Pick any S_1 such that $q_{i_1}(S_1) > 0$. Let $g_2 = g^{(i_1, S_1)}$. Similarly, pick $i_2 \in \text{argmax}_{i \in I(g_2)} d(i, i^*; g_2)$. Pick any S_2 such that $q_{i_2}(S_2) > 0$. Since $(\mathbf{x}, \mathbf{q})$ is efficient, $|S_1| \geq 2$, $|S_2| \geq 2$, and so on; and $I(g_1) \supsetneq I(g_2) \supsetneq \cdots$. Thus, one can repeat this process until $I(g_T) = \emptyset$, after which g_T is a pre-complete network. This proves the first part. In addition, assume that g is not circular. If g is a tree, then any induced network cannot be circular and hence g_T is not circular. If g has a cycle but not a circular network, then $\deg_{i^*}(g) = \deg_{i^*}(g_T) \geq 3$, and hence g_T cannot be circular. □

4 Non-transferability of Recognition Probabilities

We have assumed that the initial recognition probabilities are transferable. That is, when they trade their communication links, they also trade their chances of being a proposer as well. In some other environment, however, players cannot trade their recognition probabilities. With non-transferable recognition probabilities, instead of (1), we define the recognition probabilities in any state π in the following way:

$$p_i^\pi = \begin{cases} \dfrac{p_i}{\sum_{k \in N^\pi} p_k} & \text{if } i \in N^\pi \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

With non-transferable recognition probabilities, obtaining an efficient equilibrium is impossible even in circular networks.

Theorem 2. *With* non-transferable recognition probabilities*, an efficient equilibrium exists for all discount factors if and only if the underlying network is* complete*.*

Before proving the theorem, we construct an inefficient equilibrium in a four-player circular network as an example.

Example 6 (A Circular Network with Non-Transferable Recognition Probabilities)**.** Let $g = (\{1,2,3,4\}, \{12, 23, 34, 14\})$ and $p = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. If $\delta > \frac{2}{3}$, then there exists a cutoff strategy equilibrium $(\mathbf{x}, \mathbf{q})$ such that

- $x_1 = x_2 = \frac{\delta}{3(2-\delta)}$; $x_3 = x_4 = \frac{\delta}{6(2-\delta)}$;
- $q_1(\{1\}) = q_2(\{2\}) = 1$; $q_3(\{3,4\}) = q_4(\{3,4\}) = 1$.

That is, player 1 and player 2 decline to make an offer and wait for a three-player complete network induced by player 3 or player 4.

i) *Optimality:* Since recognition probabilities are not transferable, note that $x_1^{(1,\{1,2\})} = x_1^{(1,\{1,4\})} = \frac{1}{3}$ and $x_1^{(1,\{1,2,4\})} = \frac{1}{2}$. Given $\mathbf{x}$, player 1's excess surpluses are:

   - $e_1(\{1,2\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_1 + x_2) = \frac{\delta(2-3\delta)}{3(2-\delta)}$
   - $e_1(\{1,4\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_1 + x_4) = \frac{\delta(1-2\delta)}{6(2-\delta)}$
   - $e_1(\{1,2,3\}, \mathbf{x}) = \frac{1}{2}\delta - \delta(x_1 + x_2 + x_3) = \frac{\delta(1-3\delta)}{6(2-\delta)}$.

   Given $\delta > \frac{2}{3}$, since $e_1(S, \mathbf{x})$ for all $\{1\} \subsetneq S \subseteq N_1$, we have $\mathcal{D}_1(\mathbf{x}) = \{\{1\}\}$, which implies $q_1(\{1\}) = 1$. Similarly we have $q_2(\{2\}) = 1$. Now calculate player 3's excess surpluses:

   - $e_3(\{2,3\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_2 + x_3) = \frac{\delta(2-3\delta)}{3(2-\delta)}$
   - $e_3(\{3,4\}, \mathbf{x}) = \frac{1}{3}\delta - \delta(x_3 + x_4) = \frac{\delta(1-\delta)}{3(2-\delta)}$
   - $e_3(\{2,3,4\}, \mathbf{x}) = \frac{1}{2}\delta - \delta(x_2 + x_3 + x_4) = \frac{\delta(2-3\delta)}{6(2-\delta)}$.

   Given $\delta > \frac{2}{3}$, since $e_3(\{3,4\}, \mathbf{x}) > 0$ and $e_1(S, \mathbf{x}) < 0$ for any other $\{1\} \subseteq S \subseteq N_1$, we have $q_3(\{3,4\}) = 1$, and similarly $q_4(\{3,4\}) = 1$.

ii) *Consistency:* Given $(\mathbf{x}, \mathbf{q})$, calculate each player's expected payoff:

   - $u_1(\mathbf{x}, \mathbf{q}) = p_1 e_1(\{1\}, \mathbf{x}) + \delta\left((p_1 + p_2)x_1 + (p_3 + p_4)\frac{1}{3}\right)$
     $= \frac{1}{4} \cdot 0 + \delta\left(\frac{1}{2}\frac{\delta}{3(2-\delta)} + \frac{1}{2}\frac{1}{3}\right) = \frac{\delta}{3(2-\delta)} = x_1,$
   - $u_3(\mathbf{x}, \mathbf{q}) = p_3 e_3(\{3,4\}, \mathbf{x}) + \delta x_3$
     $= \frac{1}{4}\left(\frac{1}{3}\delta - 2 \cdot \frac{\delta^2}{6(2-\delta)}\right) + \frac{\delta^2}{6(2-\delta)} = \frac{\delta}{6(2-\delta)} = x_3,$

and similarly, $u_2(\mathbf{x},\mathbf{q}) = x_2$ and $u_4(\mathbf{x},\mathbf{q}) = x_4$.

The equilibrium payoff vector converges to $(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$. Furthermore, the equilibrium payoff vector is not unique even as $\delta \to 1$. Note that there exists another class of equilibrium payoff vectors which converge to $(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ or its permutations. However, there is no symmetric equilibrium. □

Now we prove Theorem 2. Since Proposition 1 still holds with non-transferable recognition probabilities, this directly proves the sufficient condition. Proposition 5 shows that an efficient equilibrium is impossible for any pre-complete network. Due to the first part of Proposition 4, then, the necessary condition is proved for any incomplete network.

Proposition 5. *Suppose that recognition probabilities are not transferable. Let g be a pre-complete. For any p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, any efficient strategy profile $(\mathbf{x},\mathbf{q})$ cannot be an equilibrium in $\Gamma = (g,p,\delta)$.*

Proof. Since g is incomplete, $N(g) \setminus D(g) \neq \emptyset$. After renaming, let $N(g) \setminus D(g) = \{1,\cdots,L\}$. For each $1 \le \ell \le L$, take any S_ℓ such that $q_\ell(S_\ell) > 0$. Since $(\mathbf{x},\mathbf{q})$ is efficient, $S_\ell \in \mathcal{C}_\ell$. Due to Proposition 1, we have $x_\ell^{(\ell,S_\ell)} = p_\ell^{(\ell,S_\ell)}$. Since non-transferable recognition probabilities are assumed, we have

$$x_\ell^{(\ell,S_\ell)} = p_\ell^{(\ell,S_\ell)} = \frac{p_\ell}{\sum_{j\in N\setminus S_\ell} p_j + p_\ell} = \sum_{j\in S_\ell} p_j - \frac{\left(\sum_{j\in N\setminus S_\ell} p_j\right)\left(\sum_{j\in S_\ell\setminus\{\ell\}} p_j\right)}{\sum_{j\in N\setminus S_\ell} p_j + p_\ell}. \quad (15)$$

In addition, since S_ℓ-formation is optimal for each ℓ, it must be $e_\ell(S_\ell,\mathbf{x}) \ge e_\ell(\{\ell\},\mathbf{x}) = 0$, or equivalently, $\sum_{j\in S_\ell} x_j \le x_\ell^{(\ell,S_\ell)}$. Summing this over ℓ and plugging (15), we have

$$\sum_{\ell=1}^{L}\sum_{j\in S_\ell} x_j \le \sum_{\ell=1}^{L}\sum_{j\in S_\ell} p_j - \rho, \quad (16)$$

where

$$\rho = \sum_{\ell=1}^{L} \frac{\left(\sum_{j\in N\setminus S_\ell} p_j\right)\left(\sum_{j\in S_\ell\setminus\{\ell\}} p_j\right)}{\sum_{j\in N\setminus S_\ell} p_j + p_\ell}.$$

Note that ρ is strictly positive and does not depend on δ.

On the other hand, for any k's S-formation such that $q_k(S) > 0$ and any $j \notin S$, it must be $x_j^{(k,S)} = p_j^{(k,S)} = \frac{p_j}{\sum_{\ell\in N\setminus S} p_k + p_k} > p_j$. Then, for each $j \in N$, j's continuation payoff is

$$\begin{aligned} u_j(\mathbf{x},\mathbf{q}) &= p_j m_j(\mathbf{x}) + \delta \sum_{k\in N} p_k \left(\sum_{S\ni j} q_k(S) x_j + \sum_{S\not\ni j} q_k(S) x_j^{(k,S)}\right) \\ &\ge \delta \sum_{k\in N} p_k \left(\sum_{S\ni j} q_k(S) x_j + \sum_{S\not\ni j} q_k(S) p_j^{(k,S)}\right) \\ &> Q_j x_j + (1-Q_j) p_j. \end{aligned}$$

where $Q_j = \sum_{k \in N} p_k \sum_{S \ni j} q_k(S)$. By the consistency condition, it follows that

$$x_j > \frac{(1-Q_j)\delta p_j}{1-Q_j\delta} \tag{17}$$

Combining (16) and (17), we have,

$$\delta \sum_{\ell=1}^{L} \sum_{j \in S_\ell} \frac{(1-Q_j)p_j}{1-Q_j\delta} + \rho < \sum_{\ell=1}^{L} \sum_{j \in S_\ell} p_j. \tag{18}$$

As $\delta \to 1$, $\delta \sum_{\ell=1}^{L} \sum_{j \in S_\ell} \frac{(1-Q_j)p_j}{1-Q_j\delta}$ converges to $\sum_{\ell=1}^{L} \sum_{j \in S_\ell} p_j$. However, this contradicts to the fact that $\rho > 0$ is fixed. Thus, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, the inequality (18) is violated. □

Remark. In network-restricted unanimity games, welfare can be improved by allowing for players to trade their recognition probabilities. If the recognition probabilities are not transferable, a proposer has less incentive to form a coalition. In general characteristic function form games, however, the effect of transferability of recognition probabilities on welfare may be negative. For instance, when there exists a veto player, non-veto players may form a union, to be a new veto coalition, as a transitional state, rather than immediately forming an efficient coalition. If the recognition probabilities are transferable, then they have stronger incentive to form transitional inefficient coalitions. Thus, for a non-unanimity game, banning players from trading recognition probabilities may improve welfare. See Lee (2014) for details.

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Appendix

Proof of i) in Proposition 1.

Case 1: $|N(g)| = 2$. Let $N(g) = \{i, j\}$ and $p = (p_i, p_j)$ with $p_i + p_j = 1$. We show that a cutoff strategy profile $(\{p_1, p_2\}, \{q_i(N) = 1, q_j(N) = 1\})$ is an equilibrium by verifying player i has no profitable deviation strategy given player j's cutoff strategy. Note that player i's expected payoff from following her cutoff strategy is $p_i(1 - \delta p_j) + p_j(\delta p_i) = p_i$. First, consider player i's proposal strategy. Either making an offer with $y_j < \delta p_j$ or declining to make an offer yields an expected payoff δp_i. Making an offer with $y_j > \delta p_j$ is not profitable since the offer $y_j = \delta p_j$ will be accepted. Thus, player i cannot be better off by deviating from the given proposal strategy. Next, consider player i's response strategy.

By rejecting any offer, player i expects the payoff p_i in the next period. Thus, rejecting any offer with $y_i < \delta p_i$ is optimal. It is clear that accepting any offer with $y_i \geq \delta p_i$ is optimal. Therefore, player i has no profitable deviation strategy given player j's cutoff strategy.

Case 2: $|N(g)| > 2$. Suppose that, for any game (g', p', δ) with $|N(g')| < |N(g)|$, $(\mathbf{p}', \bar{\mathbf{q}}')$ is an equilibrium, where $\mathbf{p}' = \{\{p_i'^{\pi}\}_{i \in N^\pi}\}_{\pi \in \Pi(g')}$ and $\bar{\mathbf{q}}' = \{\{\bar{q}_i^{\pi}\}_{i \in N^\pi}\}_{\pi \in \Pi(g')}$. Note that, in such an equilibrium, for each $i \in N(g')$, player i's expected payoff is p_i'. We show that a cutoff strategy profile $\sigma = (\mathbf{p}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) by verifying player i has no profitable deviation strategy given other players cutoff strategies. Recall that if player i follows the cutoff strategy, then her expected payoff is $p_i(1-\delta) + \delta p_i = p_i$. Since all the other players except for i are supposed to play stationary strategies, it is enough to consider the proposal strategy and the response strategy of player i separately.

- *Proposal strategy:* Consider player i's proposal strategy q_i such that $q_i(S) > 0$ for some $S \subsetneq N$ instead of $\bar{q}_i$. By forming $S \subsetneq N$, player i expects $p_i^{(i,S)}$ in the subsequent game, because $(g^{(i,S)}), p^{(i,S)}, \delta)$ is a less-than-n-player game with a complete network. In order for S to form, it must be $y_j \geq \delta p_j$ for all $j \in S \setminus \{i\}$. Note also that $p_i^{(i,S)} \leq p_S$.[16] Thus, player i's proposal gain from S-formation is

$$\delta p_i^{(i,S)} - \sum_{j \in S \setminus \{i\}} y_j \leq \delta p_S - \sum_{j \in S \setminus \{i\}} \delta p_j = \delta p_i. \tag{19}$$

  On the other hand, player i's proposal gain from following $\bar{q}_i$ is

$$1 - \sum_{j \in N \setminus \{i\}} \delta p_j = (1-\delta) p_N + \delta p_i = (1-\delta) + \delta p_i. \tag{20}$$

  Since (19) is strictly less than (20), any proposal strategy which forms $S \subsetneq N$ is not optimal for i. Among proposal strategies which form N, it is clear that making an offer with $y = \delta p$ is optimal.

- *Response strategy:* Since each $j \in N \setminus \{i\}$ is supposed to play the given cutoff strategy, player i is guaranteed at least δp_i by rejecting any offer. Thus, it is optimal for i to accept any offer with $y_i \geq \delta p_i$ and to reject any offer with $y_i < \delta p_i$. □

Proof of ii) in Proposition 1.

The statement is true for $|N(g)| = 2$ from the proof of Proposition 1. As an induction hypothesis, suppose that the statement is true for any game with less-than-n-player games

[16] With transferable recognition probabilities, it holds with equality. With non-transferable recognition probabilities, the inequality is strict.

and now consider a game $\Gamma = (g, p, \delta)$ with $|N(g)| = n$. Due to Lemma 1, only cutoff strategy equilibria are considered. Suppose that there exists a cutoff strategy equilibrium $(\mathbf{x}, \mathbf{q})$.

Case 1: Suppose that $q = \bar{q}$. For each $i \in N$, since $\sum_{k \in N} p_k \sum_{S \subseteq N} q_k(S) \mathbb{1}(i \in S) = 1$, we have

$$u_i(\mathbf{x}, \bar{\mathbf{q}}) = p_i (1 - \delta x_N) + \delta x_i. \tag{21}$$

Due to consistency, we obtain $u_i(\mathbf{x}, \bar{\mathbf{q}}) = x_i$ and $x_N = 1$. Plugging them into (21), we have $x_i = p_i$. Thus, for any cutoff equilibrium involving maximum coalition formation strategies $\bar{q}$ yields a payoff vector p.

Case 2: Suppose that there exists i who plays a non-maximum coalition formation strategy so that $q_i(S) > 0$ with $S \subsetneq N$. This implies that

- $x_N = u_N(\mathbf{x}, \mathbf{q}) < 1$; and
- there exists $S \subsetneq N$ such that $i \in S$ and $e_i(S, \mathbf{x}) \geq e_i(N, \mathbf{x})$.

Thus for each $i \in S$, we have

$$\delta x_i^{(i,S)} - \delta x_S \geq 1 - \delta x_N > 1 - \delta.$$

By the induction hypothesis, the inequality implies

$$\delta x_S + 1 < \delta p_S + \delta. \tag{22}$$

On the other hand, by letting $Q_j = \sum_{k \in N} p_k \sum_{S \subseteq N} q_k(S) \mathbb{1}(j \in S)$, for each $j \in S$, we have

$$\begin{aligned} x_j = u_j(\mathbf{x}, \mathbf{q}) &\geq p_j (1 - \delta x_N) + \delta(Q_j x_j + (1 - Q_j) p_j) \\ &> p_j (1 - \delta) + \delta(Q_j x_j + (1 - Q_j) p_j) \\ &= p_j + \delta Q_j (x_j - p_j). \end{aligned} \tag{23}$$

Rearranging the terms, (23) yields $x_j > p_j$ for all $j \in S$. However, this contradicts to (22) for all δ. □

Proof of Proposition 2.

Define $\eta(g) = \lfloor |N(g)|/2 \rfloor - 1$. If g is circular and $\eta(g) = 0$, then g must be a three-player circle, which is complete. Proposition 1 proves this case. As an induction hypothesis, suppose that, for all circular network g' such that $\eta(g') < m$, a cutoff strategy profile $(\mathbf{x}', \bar{\mathbf{q}}')$ is an equilibrium for (g', p', δ), where $\mathbf{x}' = \{\{\delta^{\eta(g'^{\pi})} p_i'^{\pi}\}_{i \in N(g'^{\pi})}\}_{\pi \in \Pi(g')}$. Now we show that a cutoff strategy profile $(\mathbf{x}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) with a circular network g and $\eta(g) = m$, where $\mathbf{x} = \{\{\delta^{\eta(g^\pi)} p_i^\pi\}_{i \in N(g^\pi)}\}_{\pi \in \Pi(g)}$. Take any $i \in N$ and let $N_i = \{i, j, k\}$. We verify the equilibrium conditions for player i.

i) *Optimality:* After i's maximum coalition formation, the active players face a game with a circular network g' and $\eta(g') = m - 1$. Due to the induction hypothesis, since $x_i^{(i,\{i,j,k\})} = \delta^{m-1}(p_i + p_j + p_k)$, we have

$$\begin{aligned} e_i(\{i,j,k\}, \mathbf{x}) &= \delta^m(p_i + p_j + p_k) - \delta(x_i + x_j + x_k) \\ &= \delta^m(p_i + p_j + p_k) - \delta(\delta^m p_i + \delta^m p_j + \delta^m p_k) \\ &= \delta^m(1 - \delta)(p_i + p_j + p_k). \end{aligned} \tag{24}$$

Suppose i decline to make an offer, that is i forms $\{i\}$. Since $e_i(\{i\}, \mathbf{x}) = 0$ is strictly less than (24), i's $\{i\}$-formation is not optimal. Suppose i forms $\{i, j\}$. Note that

$$x_i^{(i,\{i,j\})} = \begin{cases} \delta^{m-1}(p_i + p_j) & \text{if } |N(g)| \text{ is even,} \\ \delta^m(p_i + p_j) & \text{if } |N(g)| \text{ is odd.} \end{cases}$$

Thus, we have

$$e_i(\{i,j\}, \mathbf{x}) \leq \delta^m(p_i + p_j) - \delta(x_i + x_j) = \delta^m(1 - \delta)(p_i + p_j),$$

which is strictly less than (24), and hence i's S-formation with $|S| = 2$ is not optimal.

ii) *Consistency:* Since all the players play maximum coalition formation strategies, player i's continuation payoff is:

$$\begin{aligned} u_i(\mathbf{x}, \mathbf{q}) &= p_i e_i(\{i,j,k\}, \mathbf{x}) + \delta \left((p_i + p_j + p_k) x_i + \sum_{\ell \in N \setminus \{i,j,k\}} p_\ell x_i^{(\ell, N_\ell)} \right) \\ &= p_i \delta^m (1 - \delta)(p_i + p_j + p_k) + \delta(p_i + p_j + p_k) x_i + \delta(1 - (p_i + p_j + p_k))\delta^{m-1} p_i. \end{aligned}$$

Since $u_i(\mathbf{x}, \mathbf{q}) = x_i$, rearranging the terms, we have

$$(1 - \delta(p_i + p_j + p_k)) x_i = p_i \delta^m (1 - \delta)(p_i + p_j + p_k) + (1 - (p_i + p_j + p_k))\delta^m p_i,$$

which yields $x_i = \delta^m p_i$. $\square$