Networks of Many Public Goods with Non-Linear Best Replies

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Abstract

We model a bipartite network in which links connect agents with public goods. Agents play a voluntary contribution game in which they decide how much to contribute to each public good they are connected to. We show that the problem of finding a Nash equilibrium can be posed as a non-linear complementarity one. The existence of an equilibrium point is established for a wide class of individual preferences. We then find a simple sufficient condition, on network structure only, that guarantees the uniqueness of the equilibria, and provide an easy procedure for building networks that respects this condition.

Keywords: bipartite graph, public good, Nash equilibrium, non-linear complementarity problem.

JEL: C72, D85, H41.

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1 Introduction

In many social or geographic systems, multiple collective goods are produced voluntarily, simultaneously, and at different scales (Olson, 1965). Take for example an irrigated perimeter in which a farmer simultaneously experiments with a new irrigation technology and a new variety of crop. The information obtained from the new irrigation technology may be of interest to one group of farmers, and the results obtained from the new crop variety to another. Similarly, consider a consumer who experiences several new products, where in each case the experience may benefit a specific part of his familial and friendship relationships. Alternatively, a municipality may introduce a number of resource conservation programs at the same time. Water conservation programs may benefit municipalities within the same river basin, energy conservation programs may be advantageous to municipalities near the same production site, and soil conservation programs can have positive effects on neighboring municipalities, for example.

There exists no theoretic model to analyze the provision of many different public goods in social or geographic networks as exemplified above. However, the principle is simple: n agents must choose whether they contribute or not to m local public goods, but these agents interact only with their “ neighbors ”, in other words, there are local network relationships between the agents for each public good. Consideration of the structure of these relationships raises some interesting questions. Here we focus on two of the more important ones. How concave must the utility functions be to guarantee the existence of an equilibrium point? How is the uniqueness of the equilibrium related to the structure of the network?

To address these questions, we model a bipartite network in which links connect agents with public goods.[1] We investigate the voluntary contribution game in which agents decide how much to contribute to each public good to which they are connected. The agents receive benefits from their own and their neighbors' contributions according to a concave benefit function.[2] The cost of the contribution to each agent is a convex function of the total contribution from the agent.[3] Within this framework, we show that the

[1]Bipartite networks have previously been used, for instance, to model economic exchange when buyers have relationships with sellers (Kranton and Minehart, 2001; Corominas-Bosch, 2004), in labor market matching problems (Bóna, 2006), and in the tragedy of the commons where there are multiple commons (Ilkiliç, 2011).

[2]The assumption of concave benefits is familiar in network games in which one public good is provided (Bloch and Zenginobuz, 2007; Bramoullé and Kranton, 2007; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014; Rébillé and Richefort, 2014). In general, this means that the value of the public good has a physical restriction.

[3]The assumption of convex costs suggests the presence of a private good and a bud-

search for an equilibrium may be posed as a *non-linear complementarity problem* (Cottle, 1966; Karamardian, 1969, 1972; Kolstad and Mathiesen, 1987).

We herein contribute to two main areas of research. First, we study the voluntary and simultaneous provision of two or more public goods. Much of the work in this field has been concerned with neutrality problems (Kemp, 1984; Bergstrom et al., 1986; Cornes and Itaya, 2010), problems of equilibrium existence (Bergstrom et al., 1986; Cornes and Itaya, 2010), and efficiency problems (Cornes and Schweinberger, 1996; Cornes and Itaya, 2010). We extend the basic model of two or more public goods to a network of agents and public goods.[4] In other words, we consider a game involving the provision of many public goods, in which the agents have multidimensional and heterogeneous strategy spaces. Given such a game, we show how the existence of a unique equilibrium is conditioned by the shapes of the individual preferences and the architecture of the network.

The second related area of literature is the analysis of network games with strategic substitutes. This class of games, pioneered among others by Ballester et al. (2006), encompasses various well known games.[5] Under complete information[6], a uniqueness condition that depends on network structure only is established for three cases: linear best responses and unipartite network (Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014), linear best responses and bipartite network (Ilkiliç, 2011), and non-linear best responses and unipartite network (Rébillé and Richefort, 2014). Here we also study a fourth case, of non-linear best responses and bipartite network, which generalizes the three other cases. Using techniques borrowed from non-negative matrix theory, we obtain a uniqueness condition that depends only on the structure of the graph.

In this paper, we are concerned with the existence and uniqueness of a pure-strategy Nash equilibrium (henceforth, PSNE) in a network game involving the provision of many public goods. In Section 2, we define the voluntary contribution game. In Section 3, the existence of a PSNE is established by requiring the appropriate shape in the individual preferences.

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getary constraint (see, e.g., Bergstrom et al., 1986; Bramoullé and Kranton, 2007).

[4]The basic model of two or more public goods is a special case of our model when the network is complete and the substitutability between contributions is perfect.

[5]Network games of public good provision belong to the class of games of strategic substitutes and positive externalities (see, e.g., Bramoullé and Kranton, 2007; Galeotti et al., 2010). Network games of Cournot competition and common property resources can be defined as games of strategic substitutes and negative externalities (see, e.g., Ilkiliç, 2011; Bramoullé et al., 2014).

[6]See Galeotti et al. (2010) for the analysis of network games under incomplete information.

In Section 4, we show that the voluntary contribution game admits a unique PSNE whenever the bipartite network is sufficiently sparse. In Section 5, we apply our results to networks in which the number of public goods equals the number of agents. Section 6 concludes. All the proofs are relegated to the Appendix.

2 A Model

Consider a model where there are m public goods $p_1, \ldots, p_m$ and n agents $a_1, \ldots, a_n$. They are embedded in a network that links agents with public goods. We represent the network as a *bipartite graph*.[7]

An undirected bipartite graph $g = \langle P \cup A, L \rangle$ consists of a set of nodes formed by public goods $P = \{p_1, ..., p_m\}$ and agents $A = \{a_1, ..., a_n\}$, and a set of links L, each link connecting an agent with a public good. A link between a_i and p_j is denoted as ij.[8] We say that an agent a_i is *connected* to a public good p_j if there is a link between a_i and p_j. We will assume that an agent can choose whether or not to contribute to a public good if and only if he is connected to it. Let $r(g)$ be the number of links in L.

Given a graph g, we will denote $N_g(p_j)$ to be the set of agents connected to p_j, i.e.,

$$N_g(p_j) = \{a_i \in A \text{ such that } ij \in L\},$$

and similarly $N_g(a_i)$ is the set of public goods to which a_i is connected, i.e.,

$$N_g(a_i) = \{p_j \in P \text{ such that } ij \in L\}.$$

Then, $\sum_{a_i \in A} |N_g(a_i)| = \sum_{p_j \in P} |N_g(p_j)| = |L| = r(g)$. For all a_i, we note $r_i(g) = |N_g(a_i)|$ and for all p_j, $r^j(g) = |N_g(p_j)|$. We will assume, without loss of generality, that each agent is connected to at least one public good and vice versa, i.e., $r_i(g)$ and $r^j(g)$ are in $\mathbb{N}^*$ for all a_i and for all p_j.[9]

We now define the column vector that shows the contributions flowing at each link in L. Given a graph g, let $\mathbf{x}_g$ be the column vector of contribu-

[7] Some of the basic notation introduced in this section is borrowed from Corominas-Bosch (2004) and Ilkiliç (2011).

[8] To avoid confusion, and because the network is undirected, we will respect the following rule in the notation of a link: the first small letter in italics always refers to an agent and the second refers to a public good.

[9] In general, a public good is provided by at least two agents. Up to Section 5, we will implicitly adopt this definition. However, because our results hold even if some public goods are provided by a single agent, we only need to impose that $r^j(g) \geq 1$ for all $p_j \in P$. See Section 5 for a discussion.

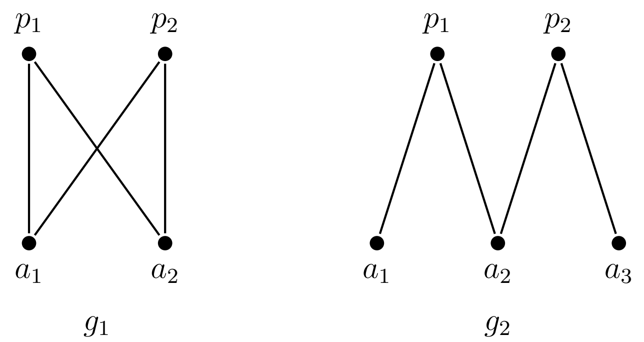

Figure 1: Networks with two public goods.

tions.[10] Hence, $\mathbf{x}_g$ is the *link by link profile of contributions* and has size $r(g)$. In the vector $\mathbf{x}_g$, the links are sorted in lexicographic order: the contribution x_{ij} is listed above the contribution x_{kl} when $i < k$ or when $i = k$ and $j < l$. For the graphs g_1 and g_2 given in Figure 1,

$$\mathbf{x}_{g_1} = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{g_2} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{22} \\ x_{32} \end{pmatrix}.$$

For a given graph g, the utility function of agent a_i is $U_i(\mathbf{x}_g)$. We will assume that the utility functions are additively separable into concave benefit and convex cost functions, all defined on $\mathbb{R}_+$ and all continuously differentiable. For a given $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$,

$$U_i\left(\mathbf{x}_g\right) = \sum_{p_j \in N_g(a_i)} b_{ij}\left(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}\right) - c_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}\right).$$

The first term is the (concave) benefit b_{ij} received from a public good p_j and summed over the public goods to which a_i is connected. The parameter $\lambda_{kj}^i \geq 0$ reflects the intensity of the positive externality received by agent a_i from agent a_k's contribution to public good p_j.[11] The second term is the (convex) cost c_i incurred by a_i. The utility function, although separable in terms of costs and benefits, is not separable with respect to each public good.

[10] All vectors considered in this paper are column vectors and are denoted by lowercase bold letters. We reserve the use of uppercase bold letters for matrices.

[11] Hence, λ_{kj}^i denotes the degree of substitutability between contribution x_{ij} and contribution x_{kj}, from the point of view of agent a_i (i.e., in general, $\lambda_{kj}^i \neq \lambda_{ij}^k$).

In particular, the marginal utility from x_{ij} does depend on the contributions by a_i to public goods other than p_j. For example in graph g_1, the contribution by agent a_1 to public good p_1 depends on his contribution to the other public good p_2.

Consider the following *voluntary contribution game*. Given a graph g, each agent a_i maximizes his utility function with respect to x_{ij} constrained to be non-negative for all $p_j \in N_g(a_i)$. The set of players is therefore the set of agents $A = \{a_1, \dots, a_n\}$, and the strategy space of agent a_i is $(\mathbf{x}_g)_i \in \mathbb{R}_+^{r_i(g)}$. For a contribution profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$, each agent a_i earns payoffs $U_i(\mathbf{x}_g) \in \mathbb{R}$. We analyze the existence and the uniqueness of the PSNE when the individual decisions are simultaneous.

3 Equilibrium Existence

In network games with strategic substitutes, the question of existence of a PSNE has received little attention. This is because individual preferences are generally specified such that best response functions are piece-wise linear, regardless of whether the agents' strategy space is uni- (Bramoullé and Kranton, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014) or multidimensional (Ilkiliç, 2011). In this section, the existence of a PSNE in a network game of strategic substitutes (i.e., the voluntary contribution game) is established when the strategy spaces of the agents are multidimensional and heterogeneous, and the set of best response functions define non-linear mappings.

Let μ_{ij} be the Karush-Kuhn-Tucker's multiplier associated with the constraint $x_{ij} \geq 0$. For all links $ij \in L$, the first order conditions are given by

$$b'_{ij}\left(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj}\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}\right) + \mu_{ij} = 0$$

with

$$\mu_{ij} x_{ij} = 0, \quad \mu_{ij} \geq 0.$$

We then deduce that all PSNEs admitted by the voluntary contribution game are solutions to a non-linear complementarity problem.[12,13]

[12] Inequalities between vectors implie inequalities between components. The superscript T denotes the transpose of a vector or a matrix.

[13] The complementarities in the network are between the contributions, which are either *strategic substitutes* or *complements*. For example in the complete graph g_1 (Fig. 1), x_{11} and x_{21} are *strategic substitutes*. They both participate in the provision of p_1. The contribution from one agent decreases the marginal benefit from p_1. This in turn decreases

Property 1. *Given a graph g, a profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$ is a PSNE of the voluntary contribution game if and only if $\mathbf{x}_g$ satisfies*

$$\mathbf{x}_g \geq \mathbf{0}, \quad \mathbf{b}'\left(\mathbf{D}_g\mathbf{x}_g\right) - \mathbf{c}'\left(\mathbf{M}_g\mathbf{x}_g\right) \leq \mathbf{0}, \quad {\mathbf{x}_g}^{\mathsf{T}}\left[\mathbf{b}'\left(\mathbf{D}_g\mathbf{x}_g\right) - \mathbf{c}'\left(\mathbf{M}_g\mathbf{x}_g\right)\right] = 0,$$

where for all links ij, $(\mathbf{b}'(\mathbf{D}_g\mathbf{x}_g))_{ij} = b'_{ij}(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj})$ and $(\mathbf{c}'(\mathbf{M}_g\mathbf{x}_g))_{ij} = c'_i(\sum_{p_j \in N_g(a_i)} x_{ij})$.

For any graph g, the columns and the rows in $\mathbf{D}_g$ and $\mathbf{M}_g$ are the links in g. In both matrices, the links are classified in the same order as in $\mathbf{x}_g$: the rows (resp. the columns) are sorted such that the link ij is listed above (resp. to the left of) the link kl when $i < k$ or when $i = k$ and $j < l$. Then, $\mathbf{D}_g = [d_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$d_{ij,kl} = \begin{cases} 1, & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ \lambda^i_{kl}, & \text{for } ij, kl \in L \text{ s.t. } i \neq k \text{ and } j = l; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } j \neq l. \end{cases}$$

We call $\mathbf{D}_g$ the *matrix of peer influences.* For example, let us take $\mathbf{D}_{g_1}$ and $\mathbf{D}_{g_2}$.

$$\mathbf{D}_{g_1} = \begin{pmatrix} 1 & 0 & \lambda^1_{21} & 0 \\ 0 & 1 & 0 & \lambda^1_{22} \\ \lambda^2_{11} & 0 & 1 & 0 \\ 0 & \lambda^2_{12} & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{g_2} = \begin{pmatrix} 1 & \lambda^1_{21} & 0 & 0 \\ \lambda^2_{11} & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda^2_{32} \\ 0 & 0 & \lambda^3_{22} & 1 \end{pmatrix}.$$

Therefore, $\mathbf{D}_g$ will generally be asymmetric, while $\mathbf{M}_g$ is symmetric by construction. More precisely, $\mathbf{M}_g = [m_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$m_{ij,kl} = \begin{cases} 1, & \text{for } ij, kl \in L \text{ s.t. } i = k; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } i \neq k. \end{cases}$$

We call $\mathbf{M}_g$ the *matrix of personal influences.* For example, let us take $\mathbf{M}_{g_1}$ and $\mathbf{M}_{g_2}$.

$$\mathbf{M}_{g_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{g_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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the incentive of the other agent to participate in the provision of p_1. Moreover, x_{21} and x_{22} are also *strategic substitutes*. They both come from a_2. The contribution to one public good increases the marginal cost incurred by a_2. This in turn decreases the incentive of a_2 to participate in the other public good. So x_{11} and x_{21}, as well as x_{21} and x_{22}, are *strategic substitutes*. This makes x_{11} and x_{22} *complements*.

The structure of any graph g is characterized by both $\mathbf{M}_g$ and $\mathbf{D}_g$. We will make use of these matrices in the next section, for the uniqueness problem. Now, for the existence of a solution to the voluntary contribution game, we have the following hypothesis.

Assumption 1 (Technical assumptions)**.**

(A1.1) For all links ij and for all agents a_i, $b'_{ij}(0) - c'_i(0) > 0$.

(A1.2) For all links ij and for all agents a_i, $b'_{ij}(\infty) - c'_i(\infty) < 0$.

(A1.3) For all links ij and for all agents a_i, b_{ij} and c_i are twice continuously differentiable, with b_{ij} strictly concave and c_i convex.

Consideration of Property 1 makes these technical assumptions very intuitive. If A1.1 is not satisfied, then agent a_i will not provide any contribution to public good p_j, and link ij can be ignored. If A1.2 is not satisfied, then agent a_i's optimization problem with respect to his contribution to public good p_j has no solution. A1.3 reflects the convexity of preferences. In other words, A1.1, A1.2 and A1.3 guarantee that each best response defines a continuous function from a compact and convex set to itself. Then, we can rely on Brouwer fixed-point theorem to establish the following result.

Theorem 1 (Existence Theorem)**.** *Given a graph g, the voluntary contribution game admits a PSNE whenever Assumption 1 is satisfied.*

This result generalizes Bergstrom et al. (1986)'s existence result to a network of agents and public goods. It also extends Rébillé and Richefort (2014)'s existence result to the multidimensional case. Furthermore, when the benefit and cost functions are quadratic (as, e.g., in Ilkiliç, 2011), it can be shown that the technical assumptions are always fulfilled.

Corollary 1. *Let the benefit function of link ij and the cost function of agent a_i be such that*

$$b_{ij}(x) = \alpha_{ij}x - \frac{\eta}{2}x^2 \quad \textit{and} \quad c_i(x) = \frac{\delta_i}{2}x^2$$

for $x \in \mathbb{R}_+$, where $\alpha_{ij}, \eta, \delta_i > 0$.[14] *Given a graph g, the voluntary contribution game always admits a PSNE.*

[14]In that case, a profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$ is a PSNE of the voluntary contribution game if and only if $\mathbf{x}_g$ satisfies

$$\mathbf{x}_g \geq \mathbf{0}, \quad \boldsymbol{\alpha}_g - (\eta\mathbf{D}_g + \mathbf{C}_g\mathbf{M}_g)\,\mathbf{x}_g \leq \mathbf{0}, \quad \mathbf{x}_g^{\mathsf{T}}\left[\boldsymbol{\alpha}_g - (\eta\mathbf{D}_g + \mathbf{C}_g\mathbf{M}_g)\,\mathbf{x}_g\right] = 0,$$

4 Equilibrium Uniqueness

We now establish a sufficient condition for a unique PSNE to the voluntary contribution game. This question has been studied in detail for the case of a single public good. Several conditions have been established, whether the best replies are linear (see, e.g., Bloch and Zenginobuz, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014) or non-linear (see, e.g., Rébillé and Richefort, 2014). However, the more realistic case of several public goods has received much less attention. When there are two or more public goods and the best replies are non-linear, we shall establish the uniqueness of the equilibria using diagonally dominant matrices.

Definition 1 (Hadamard)**.** A real matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is said to be *row diagonally dominant* (rdd) if

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|, \quad i = 1, \dots, n,$$

and *strictly row diagonally dominant* (srdd) if a strict inequality holds for all i.

Square matrices with dominant diagonals play a key role in mathematical economics. This is essentially due to the fact that all the principal minors of a srdd matrix with positive diagonal entries are positive (Berman and Plemmons, 1994, Theorem (2.3) p. 134). All diagonally dominant matrices therefore fall within the scope of the well known Hawkins-Simon condition that guarantees the existence of a solution in the input-output system. They also serve as a basis for establishing the stability of a competitive market (see, e.g., McKenzie, 1960).[15]

With this definition in mind, we impose conditions on the structure of the network *only*, under which the voluntary contribution game admits at most one PSNE. In contrast with classic results on the uniqueness of solutions

where $\boldsymbol{\alpha}_g = [\alpha_{ij}]_{1\times r(g)}$ and $\mathbf{C}_g = [c_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$c_{ij,kl} = \begin{cases} \delta_i, & \text{for } ij, kl \in L \text{ s.t. } ij = kl, \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl. \end{cases}$$

In $\boldsymbol{\alpha}_g$, the links (i.e., the rows) are sorted as in $\mathbf{x}_g$. In $\mathbf{C}_g$, the links (i.e., the rows and the columns) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$. Ilkiliç (2011) studied a particular version of this problem, where $\alpha_{ij} = \alpha$ for all $ij \in L$, $\eta = 2\beta$, $\delta_i = \gamma$ for all $a_i \in A$ and $\lambda^i_{kj} = 1/2$ for all $ij, kj \in L$, $k \neq i$.

[15] In this literature, the usual definition of a diagonal dominant matrix is slightly more general than that adopted herein (see McKenzie, 1960, p. 47).

to the non-linear complementarity problem, we impose no conditions on the mapping (here, the vector-valued function of marginal utilities).[16] We reason by contradiction and obtain the following result.

Theorem 2 (Uniqueness Theorem)**.** *Let Assumption 1 be satisfied. Given a graph g, the voluntary contribution game admits a unique PSNE whenever*

$$r_i(g) \leq 2, \quad i = 1, \ldots, n,$$

and

$$\sum_{a_k \in N_g(p_j) \backslash \{a_i\}} \lambda^i_{kj} < 1$$

for all $ij \in L$.

For the proof, we show that the non-linear complementarity problem associated with the voluntary contribution game (see Property 1) admits at most one solution whenever $\mathbf{M}_g$ is rdd and $\mathbf{D}_g$ is srdd. Due to its Boolean nature, the matrix of personal influences $\mathbf{M}_g$ is rdd if and only if each agent is connected to at most two public goods, i.e.,

$$r_i(g) \leq 2, \quad i = 1, \ldots, n.$$

This does not mean, however, that there should be a maximum of two public goods in the graph; this depends on the number of connections per agent. For example, in the four graphs given in Figure 2, there are three public goods but the structure of graphs g_3 and g_4 comply with the first assumption of Theorem 2 (i.e., $\mathbf{M}_{g_3}$ and $\mathbf{M}_{g_4}$ are rdd). By contrast, graphs g_5 and g_6 do not, because each of these graphs contains at least one agent with three connections.

Furthermore, we can always add a new public good to a graph respecting the condition $r_i(g) \leq 2$. If there exist two agents with only one connection, the addition of a new public good can be achieved simply by creating a new connection from these agents to the new public good. Otherwise, the addition of a new public good requires the introduction of new agents in the graph. Graphs $g_{3'}$ and $g_{4'}$ given in Figure 3 illustrate these two situations.

[16]For instance, in contrast with Karamardian (1969), we do not assume that the vector-valued function of marginal utilities is (strictly) monotonic. Moreover, unlike Kolstad and Mathiesen (1987), we do not exclude the possibility that at PSNE, an agent may be just at the margin of choosing whether to contribute to a given public good. See Facchinei and Pang (2003) for a survey of sufficient conditions on the mapping for the uniqueness of solutions to the non-linear complementarity problem.

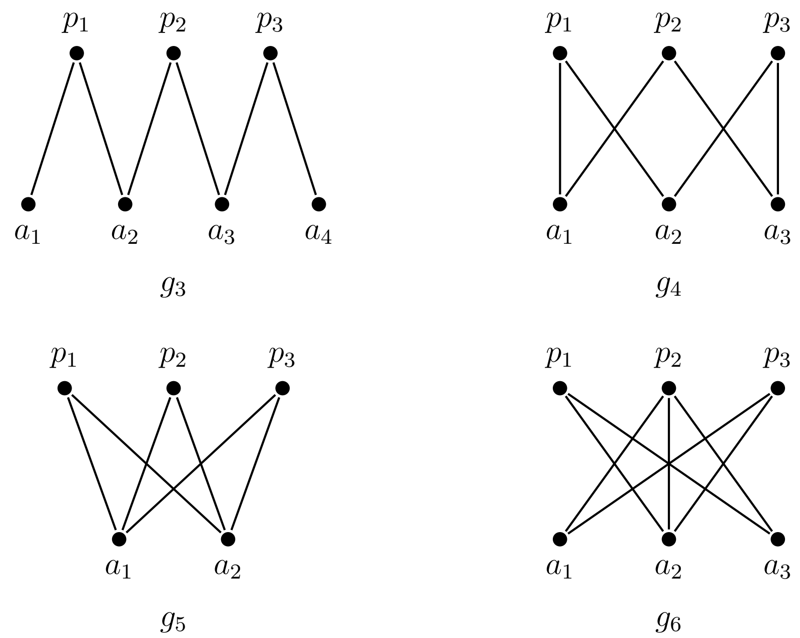

Figure 2: Networks with three public goods. Graphs g_3 and g_4 support the first condition of Theorem 2, while g_5 and g_6 do not.

The matrix of peer influences $\mathbf{D}_g$ is srdd if and only if each agent does not benefit too much from his peers, i.e.,

$$\sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda^i_{kj} < 1$$

for all $ij \in L$. Geometrically, these conditions on $\mathbf{M}_g$ and $\mathbf{D}_g$ imply that the voluntary contribution game admits a unique PSNE whenever the bipartite network is *sufficiently sparse*. If one agent is connected to three (or more) public goods[17] or if peer influences are too high, then the voluntary contribution game might admit multiple PSNEs. When there are only two public goods and peer influences are identical for a given public good, we have the following stronger results.

Corollary 2. *Let Assumption 1 be satisfied, and let $\lambda^i_{kj} = \lambda_j$ for all $ij, kj \in L$, $k \neq i$.*

(i) Given a graph g where $P = \{p_1, p_2\}$, the voluntary contribution game

[17]This happens particularly when the number of public goods exceeds the number of agents. See Section 5 for a discussion.

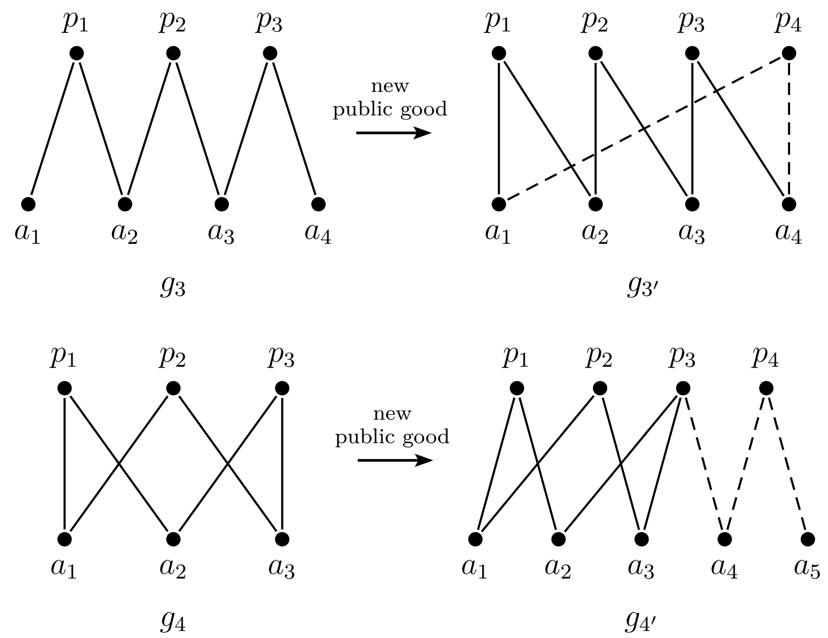

Figure 3: Adding public goods whilst respecting the first condition of Theorem 2.

admits a unique PSNE whenever

$$\lambda_j < \frac{1}{r^j(g) - 1}, \quad j = 1, 2.$$

(ii) Given a complete *graph g where $P = \{p_1, p_2\}$, the voluntary contribution game admits a unique PSNE whenever*

$$\lambda_j < \frac{1}{n - 1}, \quad j = 1, 2.$$

For instance, graph g_2 falls within the scope of part (i) of Corollary 2, while part (ii) applies to graph g_1.

5 Application: the case $n = m$

We now apply our results to networks in which the number of agents n equals the number of public goods m. We begin by specifying the nature of the public goods we consider in this section.

Definition 2. A public good p_j is a *collective good* if at least two agents participate in its provision, i.e., $r^j(g) \geq 2$. A public good p_j is an *individual good* if only one agent participates in its provision, i.e., $r^j(g) = 1$.

The structure of a graph $g = \langle P \cup A, L \rangle$ indicates which public goods are collectively produced, and which are produced individually. Let C and I denote the sets of collective and individual goods, respectively. Then, the set of public goods is the union of the sets of collective and individual goods, $P = C \cup I$. Let c denote the number of collective goods. Then $c = |C|$ and $0 \leq c \leq m$.

The first condition of Theorem 2 entails that, in a graph admitting a unique PSNE, no agent should have three or more connections, i.e., $r_i(g) \leq 2$ for all $a_i \in A$. Then necessarily, the number of public goods has an upper bound given by twice the number of agents, $m \leq 2n$. In particular, $m = 2n$ if and only if each agent is connected to two individual goods, i.e., $m = |I|$ or $c = 0$. In the same vein, the number of collective goods has an upper bound given by the number of agents, $c \leq n$. Indeed, for each collective good $p_j \in C$, we have $r^j(g) \geq 2$, so $r(g) = \sum_j r^j(g) \geq 2c$. Moreover, the number of links cannot exceed twice the number of agents, i.e., $r(g) = \sum_i r_i(g) \leq 2n$, because each agent has at most two connections. Hence $c \leq n$, or in other words, the number of collective goods in a graph admitting a unique PSNE cannot exceed the number of agents.

We now focus on the case in which the number of collective goods equals the number of agents, $c = n$. So $r^j(g) = r_i(g) = 2$ for all p_j and for all a_i. A straightforward way of building a bipartite graph with $c = n$ is to consider the *circular bipartite graph* $\mathcal{C}_{n,n}$ or the *circular (unipartite) graph* $\mathcal{C}_n$ over the set of agents.

For $i = 1, \ldots, n-1$, agent a_i is connected to public goods p_i and p_{i+1} and agent a_n is connected to public goods p_n and p_1. The collective good p_j is therefore provided by agents a_j and a_{j-1} for $j = 2, \ldots, c$, and p_1 is provided by agents a_1 and a_n. Hence, any circular bipartite graph can be identified with a circular graph where nodes are identified with agents and links with public goods. For example, $g_{3'}$ is similar to the $c = n = 4$ circular bipartite graph $\mathcal{C}_{4,4}$, which is identified with graph $\mathcal{C}_4$ (Fig. 4).

Circular bipartite graphs provide a simple procedure for building $c = n$ bipartite graphs. Let $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ be K circular bipartite graphs with $\mathcal{C}_{n_k,n_k} = \langle P_k \cup A_k, L_k \rangle$. Then, we may build a $c_1 + \ldots + c_K = n_1 + \ldots + n_K$ bipartite graph $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K} = \langle P \cup A, L \rangle$ by *disjoint union* forming with $P = \cup_{k=1}^K P_k$, $A = \cup_{k=1}^K A_k$ and $L = \cup_{k=1}^K L_k$. For instance, two of the possible representations of the $c = n = 6$ bipartite graph are given by $\mathcal{C}_{6,6}$ and $\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$ (Fig. 5).

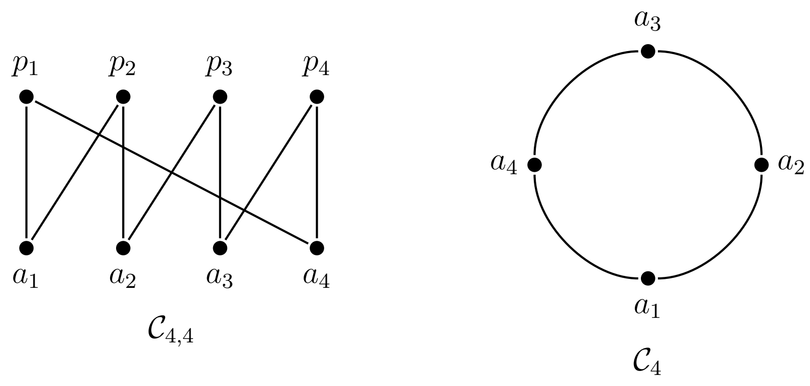

Figure 4: Circular graphs with four agents (and four public goods).

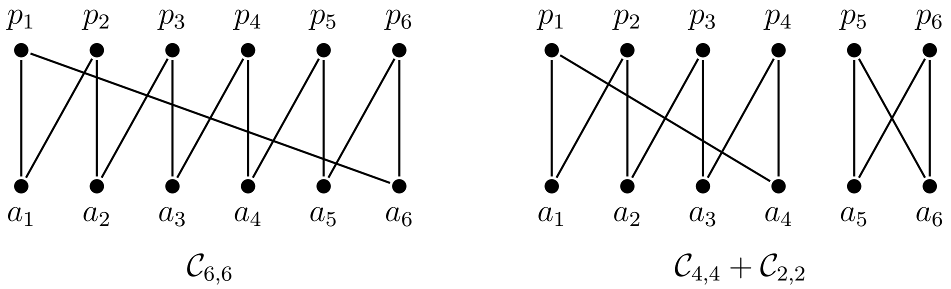

Figure 5: Two representations of the $c = n = 6$ bipartite graph.

The converse also holds, i.e., any $c = n$ bipartite graph is a disjoint union of circular bipartite graphs. That is, for any $c = n$ bipartite graph g, there are circular bipartite graphs $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ with $\sum_{k=1}^{K} n_k = n$ and $n_k \geq 2$ for all k such that $g = \mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$. This result can be formalized as follows.

Proposition 1. *Let $g = \langle P \cup A, L \rangle$ be a graph. Then, g is a $c = n$ bipartite graph if and only if there exist circular bipartite graphs $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ such that*

$$g = \mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}.$$

Moreover, the decomposition is unique.

Nevertheless, for a given $c = n$, there exist different possible decompositions into circular bipartite graphs. For instance, a $6 = 6$ bipartite graph can be obtained through $\mathcal{C}_{6,6}$, $\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$, $\mathcal{C}_{3,3} + \mathcal{C}_{3,3}$ or $\mathcal{C}_{2,2} + \mathcal{C}_{2,2} + \mathcal{C}_{2,2}$. This question is intimately related to the *partition of an integer*.[18]

Any integer n can be partitioned into sums of integers. Let $p(n)$ be the number of (unordered) partitions of n. Formally,

$$p\left(n\right)=\left|\left\{\left(n_{1},\ldots,n_{K}\right):n=n_{1}+\ldots+n_{K},n_{1}\geq\ldots\geq n_{K}\geq1,n_{k}\in\mathbb{N}\right\}\right|,$$

for $n \geq 1$. Similarly, the number of (unordered) decompositions of a circular $c = n$ bipartite graph is given by $p_2(n)$, the number of (unordered) partitions of n with classes of size at least 2,

$$p_{2}\left(n\right)=\left|\left\{\left(n_{1},\ldots,n_{K}\right):n=n_{1}+\ldots+n_{K},n_{1}\geq\ldots\geq n_{K}\geq2,n_{k}\in\mathbb{N}\right\}\right|,$$

for $n \geq 2$. The connection is made through $p_2(n) = p(n) - p(n-1)$, that is $p(n) = p(n-1) + p_2(n)$. Indeed, a partition of n either includes a class of size 1 or it does not. If the partition includes a class of size 1, then the partition without this class of size 1 is a partition of $n-1$. Otherwise, there is no class of size 1 in the partition, so each class is of size at least 2. Alternatively, we have

$$\begin{array}{rcccccccccc} p\left(n\right) & = & \left(p\left(n\right)-p\left(n-1\right)\right) & + & \ldots & + & \left(p\left(2\right)-p\left(1\right)\right) & + & p\left(1\right) \\ & = & p_{2}\left(n\right) & + & \ldots & + & p_{2}\left(2\right) & + & 1. \end{array}$$

Any partition of n may contain $l \leq n-2$ classes of size 1, thus $n-l$ remains to be shared into classes of size at least 2, and hence $p_2(n-l)$ possible partitions. Otherwise, the partition contains at least $n-1$ classes of size 1, in which case it coincides with the trivial partition into n classes of size 1. Table 1 illustrates this result when $c = n \leq 6$.

Finally, let us provide an approximation of $p_2(n)$. Hardy and Ramanujan (1917, Eq. (5.22) p. 130) establish the following asymptotic formula for $p(n)$,

$$p\left(n\right)=\exp\left\{\pi\sqrt{\frac{2n}{3}}\left(1+\epsilon_{n}\right)\right\}$$

where $\lim_n \epsilon_n = 0$, that is,

$$\ln p\left(n\right)\sim\pi\sqrt{\frac{2n}{3}}.$$

We may wonder if one can obtain a similar one for $p_2(n)$. The answer is affirmative, in fact $p_2(n)$ admits the same asymptotic formula.

[18] See, e.g., Chapter 5 in Bóna (2006) for an introduction to this problem.

Table 1: Decompositions of $c = n \leq 6$ bipartite graphs, and the 11 partitions of 6.

n	Decompositions of $c = n$ bipartite graphs	Partitions of 6
6	$\mathcal{C}_{6,6}$ $\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$ $\mathcal{C}_{3,3} + \mathcal{C}_{3,3}$ $\mathcal{C}_{2,2} + \mathcal{C}_{2,2} + \mathcal{C}_{2,2}$	6 4 + 2 3 + 3 2 + 2 + 2
5	$\mathcal{C}_{5,5}$ $\mathcal{C}_{3,3} + \mathcal{C}_{2,2}$	1 + 5 1 + 3 + 2
4	$\mathcal{C}_{4,4}$ $\mathcal{C}_{2,2} + \mathcal{C}_{2,2}$	1 + 1 + 4 1 + 1 + 2 + 2
3	$\mathcal{C}_{3,3}$	1 + 1 + 1 + 3
2	$\mathcal{C}_{2,2}$	1 + 1 + 1 + 1 + 2
1		1 + 1 + 1 + 1 + 1 + 1

Proposition 2 (Asymptotic Enumeration)**.** *For an integer $n \geq 2$, let $p_2(n)$ denote the number of unordered partitions of n with classes of size at least 2. Then,*

$$\ln p_2(n) \sim \pi \sqrt{\frac{2n}{3}}.$$

Hence, $\ln p_2(n) \sim \ln p(n)$.

Therefore, the number of decompositions of a $c = n$ bipartite graph into circular bipartite graphs approaches $\exp\{\pi\sqrt{\frac{2n}{3}}\}$ as the number of public goods (and the number of agents) approaches infinity.

6 Conclusion

We have analyzed a network game of public good provision in which there are many public goods. Under conditions on individual preferences that are as weak as possible, we show that there exists a unique PSNE whenever the bipartite network is sufficiently sparse. A simple procedure to build networks respecting the uniqueness condition is finally established for graphs in which the number of agents equals the number of public goods.

These results have been derived for the (general) case of network games with non-linear best replies and multidimensional strategy spaces. To our knowledge, all previous results on equilibrium existence for network games of the provision of one public good are special cases of our existence result. We believe, however, that the main contribution of the paper is Theorem 2, because this result is the first to provide a sufficient condition, that depends on network structure only, for the uniqueness of equilibria in network games of the provision of many local public goods. Interestingly, it applies to all games that can be studied through the same complementarity problem as that described by Property 1. This is the case, for instance, for network games of strategic substitutes and negative externalities such as the game of Cournot competition or the water extraction game (see, e.g., Okuguchi, 1983; Kolstad and Mathiesen, 1987; Ilkiliç, 2011).

Our analysis paves way for further research. Firstly, the question should be explored of whether a sharper condition for uniqueness can be obtained. In particular, does the P-matrix condition established when there is only one public good hold when there are two or more public goods? Answering this question might require the use of other algebraic techniques such as, e.g., determinantal inequalities for the product and sum of matrices. Secondly, given that we know when the equilibrium exists and is unique, it may be possible to study the structure of the equilibrium. When the best replies are linear, numerous authors on network games have expressed the equilibrium in terms of the Katz-Bonacich centrality vector (see, e.g., Ballester et al., 2006; Ballester and Calvó-Armengol, 2010). When the best replies are non-linear, the relationship between the equilibrium and the Katz-Bonacich centrality vector is less obvious, but this question nevertheless remains an important challenge. Then, it may be interesting to implement our model in an experiment, in order to test whether the behaviors conform well with theoretical predictions.

Appendix

Proof of Theorem 1. Since $b_{ij} - c_i$ is strictly concave, $b'_{ij}(0) - c'_i(0) > 0$ and $b'_{ij}(M) - c'_i(M) < 0$ for some $M > 0$, there exists a unique x^*_{ij} such that $b'_{ij}(x^*_{ij}) - c'_i(x^*_{ij}) = 0$. Moreover, x^*_{ij} is link ij's maximum.

Let $S_{-i,j} = \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj} \geq 0$ and $C_{i,-j} = \sum_{p_l \in N_g(a_i)\setminus\{p_j\}} x_{il} \geq 0$.

Agent a_i's utility is given by

$$U_i\left(\mathbf{x}_g\right) = \sum_{p_j \in N_g(a_i)} b_{ij}\left(x_{ij} + S_{-i,j}\right) - c_i\left(x_{ij} + C_{i,-j}\right)$$

By assumption, b_{ij} is strictly concave and c_i is convex, so $b'_{ij} - c'_i$ is strictly decreasing and continuous. Given $S_{-i,j}$ and $C_{i,-j}$, the best response for each link $ij \in L$ is

$$\phi_{ij}\left(S_{-i,j}, C_{i,-j}\right) = \begin{cases} \left[b'_{ij}\left(. + S_{-i,j}\right) - c'_i\left(. + C_{i,-j}\right)\right]^{-1}(0), & \\ & \text{if } b'_{ij}\left(0 + S_{-i,j}\right) - c'_i\left(0 + C_{i,-j}\right) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $b'_{ij}(. + S_{-i,j}) \leq b'_{ij}(.)$ and $c'_i(. + C_{i,-j}) \geq c'_i(.)$, we have

$$b'_{ij}\left(. + S_{-i,j}\right) - c'_i\left(. + C_{i,-j}\right) \leq b'_{ij}\left(.\right) - c'_i\left(.\right),$$

so

$$\phi_{ij}\left(S_{-i,j}, C_{i,-j}\right) \leq \phi_{ij}\left(0,0\right) = x^*_{ij}.$$

It follows that the autarkic contribution is always greater than the equilibrium contribution in a bipartite network.

We now check that the best response is continuous w.r.t. $S_{-i,j}$ and $C_{i,-j}$. Let $S_{-i,j}, C_{i,-j} \geq 0$.

1st case: $b'_{ij}(0 + S_{-i,j}) - c'_i(0 + C_{i,-j}) < 0$. Then, $\phi_{ij}(S_{-i,j}, C_{i,-j}) = 0$. Since b'_{ij} and c'_i are continuous, there exists some neighborhood V of $S_{-i,j}$ and W of $C_{i,-j}$ such that $b'_{ij}(S) - c'_i(C) < 0$ for $S \in V$ and $C \in W$. Thus, $\phi_{ij}(S, C) = 0$ for $S \in V$ and $C \in W$, so ϕ_{ij} is continuous at $S_{-i,j}$ and $C_{i,-j}$.

2nd case: $b'_{ij}(0 + S_{-i,j}) - c'_i(0 + C_{i,-j}) \geq 0$. By definition, $(x_{ij}, S_{-i,j}, C_{i,-j})$ with $x_{ij} = \phi_{ij}(S_{-i,j}, C_{i,-j})$ is a solution to the equation

$$z_{ij}\left(x, S, C\right) = b'_{ij}\left(x + S\right) - c'_i\left(x + C\right) = 0.$$

Now, we observe that

$$\frac{\partial z_{ij}}{\partial x}\left(x_{ij}, S_{-i,j}, C_{i,-j}\right) = b''_{ij}\left(x_{ij} + S_{-i,j}\right) - c''_i\left(x_{ij} + C_{i,-j}\right) < 0$$

by strict-concavity, so in accordance with the implicit function theorem, there exists some differentiable function ζ such that

$$\zeta\left(S_{-i,j}, C_{i,-j}\right) = x_{ij}$$

on some open neighborhood V of $(S_{-i,j}, C_{i,-j})$, satisfying

$$z_{ij}\left(\zeta(S,C), S, C\right) = b'_{ij}\left(\zeta(S,C) + S\right) - c'_i\left(\zeta(S,C) + C\right) = 0.$$

Thus, $\phi_{ij}(S_{-i,j}, C_{i,-j}) = x_{ij} = \zeta\left(S_{-i,j}, C_{i,-j}\right)$ on $V \ni (S_{-i,j}, C_{i,-j})$, so ϕ_{ij} is continuous at $S_{-i,j}$ and $C_{i,-j}$.

Consider the mapping

$$\begin{array}{rlcl} \Phi: & \prod_{ij\in L}\left[0, x^*_{ij}\right] & \to & \prod_{ij\in L}\left[0, x^*_{ij}\right] \\ & \mathbf{x}_g & \mapsto & \left(\phi_{ij}\left(\sum_{a_k\in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj}, \sum_{p_l\in N_g(a_i)\setminus\{p_j\}} x_{il}\right)\right)_{ij} \end{array}$$

Φ is continuous w.r.t. $\mathbf{x}_g$ since ϕ_{ij}, $\mathbf{x}_g \mapsto \sum_{a_k\in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj}$ and $\mathbf{x}_g \mapsto \sum_{p_l\in N_g(a_i)\setminus\{p_j\}} x_{il}$ are continuous for all ij. According to Brouwer fixed-point theorem, Φ admits a fixed-point $\mathbf{x}_g$ which is a PSNE of the voluntary contribution game, by construction. □

The following lemma plays an important role in establishing our uniqueness result.

Lemma 1. *Let g be a graph. For all $\mathbf{x}^1_g, \mathbf{x}^2_g$ in $\mathbb{R}^{r(g)}_+$ with $\mathbf{x}^1_g \neq \mathbf{x}^2_g$, there exists a link ij such that*

$$\left(x^1_{ij} - x^2_{ij}\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}^1_g\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}^2_g\right)\right] < 0$$

whenever $\mathbf{M}_g$ is rdd and $\mathbf{D}_g$ is srdd.

For its proof, we need to recall the class of P-matrices.

Definition 3 (Fiedler and Pták, 1962)**.** An $n \times n$ real matrix $\mathbf{A}$ is said to be a *P-matrix* if there exists k such that $x_k(\mathbf{A}\mathbf{x})_k > 0$ for all nonzero $\mathbf{x}$ in $\mathbb{R}^n$.

A srdd matrix with positive diagonal entries is a P-matrix (see Berman and Plemmons, 1994, Theorem (2.3) p. 134, M_{35} implies A_5).

Proof of Lemma 1. Let $\mathbf{x}^1_g$ and $\mathbf{x}^2_g$ be two arbitrary vectors in $\mathbb{R}^{r(g)}_+$. For each link ij, let

$$\psi_{ij}(\varepsilon) = \frac{\partial U_i}{\partial x_{ij}}\left(\varepsilon \mathbf{x}^1_g + (1-\varepsilon)\,\mathbf{x}^2_g\right).$$

Since $\mathbb{R}_+^{r(g)}$ is convex, $\varepsilon \mathbf{x}_g^1 + (1-\varepsilon)\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$ for all $0 \leq \varepsilon \leq 1$. We have

$$\psi_{ij}(1) - \psi_{ij}(0) = \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right),$$

$$\psi'_{ij}(\varepsilon) = \nabla \frac{\partial U_i}{\partial x_{ij}}(\mathbf{x}_g)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right),$$

where $\mathbf{x}_g = \varepsilon \mathbf{x}_g^1 + (1-\varepsilon)\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$ and $\nabla \frac{\partial U_i}{\partial x_{ij}}(\mathbf{x}_g)$ is the gradient of $\frac{\partial U_i}{\partial x_{ij}}$ at $\mathbf{x}_g$. Then, $\nabla \frac{\partial U_i}{\partial x_{ij}}(\mathbf{x}_g)$ is a row vector of size $r(g)$, in which the columns (i.e., the links) are sorted as in (the transpose of) $\mathbf{x}_g$. Applying the mean-value theorem on ψ_{ij}, we have

$$\psi_{ij}(1) - \psi_{ij}(0) = \psi'_{ij}(\bar{\varepsilon}_{ij}) = \nabla \frac{\partial U_i}{\partial x_{ij}}\left(\bar{\mathbf{x}}_g^{[ij]}\right)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)$$

for some $0 < \bar{\varepsilon}_{ij} < 1$, where $\bar{\mathbf{x}}_g^{[ij]} = \bar{\varepsilon}_{ij}\mathbf{x}_g^1 + (1-\bar{\varepsilon}_{ij})\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$. Thus, for each link ij,

$$\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right) = \left(\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)\right)_{ij}$$

where $\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is the $r(g) \times r(g)$ " Jacobian " matrix[19] of the marginal utilities, where the rows and the columns (i.e., the links) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$. Then, $\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is such that each row ij is given by the gradient $\nabla \frac{\partial U_i}{\partial x_{ij}}(\bar{\mathbf{x}}_g^{[ij]})$.[20]

[19] This is not the " true " Jacobian matrix, since the Jacobian matrix $\mathbf{J}_F(\mathbf{x})$ of a differentiable mapping $F: D \to \mathbb{R}^n$, where D is a closed rectangular region of $\mathbb{R}^n$, is evaluated at a given $\mathbf{x} \in D$.

[20] For example, let us take $\mathbf{J}_{U'}(\bar{\mathbf{x}}_{g_1})$ (cf. graph g_1 at Fig. 1).

$$\mathbf{J}_{U'}(\bar{\mathbf{x}}_{g_1}) = \begin{pmatrix} \nabla \frac{\partial U_1}{\partial x_{11}}\left(\bar{\mathbf{x}}_{g_1}^{[11]}\right) \\ \nabla \frac{\partial U_1}{\partial x_{12}}\left(\bar{\mathbf{x}}_{g_1}^{[12]}\right) \\ \nabla \frac{\partial U_2}{\partial x_{21}}\left(\bar{\mathbf{x}}_{g_1}^{[21]}\right) \\ \nabla \frac{\partial U_2}{\partial x_{22}}\left(\bar{\mathbf{x}}_{g_1}^{[22]}\right) \end{pmatrix} = \begin{pmatrix} \nabla \frac{\partial U_1}{\partial x_{11}}\left(\bar{\varepsilon}_{11}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{11})\mathbf{x}_{g_1}^2\right) \\ \nabla \frac{\partial U_1}{\partial x_{12}}\left(\bar{\varepsilon}_{12}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{12})\mathbf{x}_{g_1}^2\right) \\ \nabla \frac{\partial U_2}{\partial x_{21}}\left(\bar{\varepsilon}_{21}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{21})\mathbf{x}_{g_1}^2\right) \\ \nabla \frac{\partial U_2}{\partial x_{22}}\left(\bar{\varepsilon}_{22}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{22})\mathbf{x}_{g_1}^2\right) \end{pmatrix}.$$

Now, given $\frac{\partial U_i}{\partial x_{ij}}(\bar{\mathbf{x}}_g^{[ij]})$ where $ij \in L$, we observe that

$$\frac{\partial^2 U_i}{\partial x_{kl} \partial x_{ij}} \left(\bar{\mathbf{x}}_g^{[ij]}\right) = \begin{cases} b_{ij}'' \left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]} \right) - c_i'' \left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]} \right), & \\ \qquad \text{for } kl \in L \text{ s.t. } kl = ij; \\ -c_i'' \left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]} \right), & \text{for } kl \in L \text{ s.t. } k = i \text{ and } l \neq j; \\ \lambda_{kj}^i b_{ij}'' \left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]} \right), & \\ \qquad \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l = j; \\ 0, & \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l \neq j. \end{cases}$$

Hence,

$$\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right) = \mathtt{B}\left(\bar{\mathbf{x}}_g\right) \mathbf{D}_g - \mathtt{C}\left(\bar{\mathbf{x}}_g\right) \mathbf{M}_g \iff -\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right) = \mathtt{C}\left(\bar{\mathbf{x}}_g\right) \mathbf{M}_g - \mathtt{B}\left(\bar{\mathbf{x}}_g\right) \mathbf{D}_g$$

where $\mathtt{B}(\bar{\mathbf{x}}_g) = [\mathtt{b}_{ij,kl}]_{r(g) \times r(g)}$ is such that[21]

$$\mathtt{b}_{ij,kl} = \begin{cases} b_{ij}'' \left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]} \right), & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl; \end{cases}$$

and $\mathtt{C}(\bar{\mathbf{x}}_g) = [\mathtt{c}_{ij,kl}]_{r(g) \times r(g)}$ is such that

$$\mathtt{c}_{ij,kl} = \begin{cases} c_i'' \left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]} \right), & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl. \end{cases}$$

By assumption, $\mathbf{D}_g$ is srdd. Then, so is $-\mathtt{B}(\bar{\mathbf{x}}_g)\mathbf{D}_g$ since $-\mathtt{B}(\bar{\mathbf{x}}_g)$ is a diagonal matrix with positive diagonal entries (by strict-concavity of the benefit functions). In addition, $\mathbf{M}_g$ is rdd. Then, so is $\mathtt{C}(\bar{\mathbf{x}}_g)\mathbf{M}_g$ since $\mathtt{C}(\bar{\mathbf{x}}_g)$ is a diagonal matrix with nonnegative diagonal entries (by convexity of the cost functions). Thus, $-\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is a P-matrix, since it is a srdd matrix with positive diagonal entries (Berman and Plemmons, 1994). By definition, there

[21] In both $\mathtt{B}(\bar{\mathbf{x}}_g)$ and $\mathtt{C}(\bar{\mathbf{x}}_g)$, the rows and the columns (i.e., the links) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$.

exists a link ij such that

$$\left(x_{ij}^1 - x_{ij}^2\right)\left(-\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)\right)_{ij} > 0 \iff \left(x_{ij}^1 - x_{ij}^2\right)\left(\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)\right)_{ij} < 0,$$

thus,

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right)\right] < 0.$$

□

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let us assume that there are two PSNE, $\mathbf{x}_g^1 \neq \mathbf{x}_g^2$. In accordance with Property 1, for each link ij,

$$x_{ij}^\alpha \left[b'_{ij}\left(x_{ij}^\alpha + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^\alpha\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^\alpha\right)\right] = 0, \quad \alpha = 1, 2,$$

and

$$b'_{ij}\left(x_{ij}^\alpha + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^\alpha\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^\alpha\right) \leq 0, \quad \alpha = 1, 2.$$

Since $\mathbf{x}_g^1, \mathbf{x}_g^2 \geq \mathbf{0}$, for each link ij, it holds

$$x_{ij}^1 \left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \leq 0$$

and

$$x_{ij}^2 \left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \leq 0.$$

It follows that, for each link ij,

$$\begin{aligned}
& x_{ij}^1 \left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \\
+ \; & x_{ij}^2 \left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \\
- \; & x_{ij}^1 \left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \\
- \; & x_{ij}^2 \left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \leq 0,
\end{aligned}$$

thus,

$$\boxed{\mathbf{A}}:\ \left(x_{ij}^1 - x_{ij}^2\right)\left[b_{ij}'\left(x_{ij}^2 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c_i'\left(\sum_{p_j\in N_g(a_i)} x_{ij}^2\right)\right.$$
$$\left. -b_{ij}'\left(x_{ij}^1 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) + c_i'\left(\sum_{p_j\in N_g(a_i)} x_{ij}^1\right)\right] \leq 0.$$

Since $r_i(g) \leq 2$ for all $a_i \in A$, $\mathbf{M}_g$ is rdd. Moreover, $\mathbf{D}_g$ is srdd as $\sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i < 1$ for all $ij \in L$. Then, according to Lemma 1, there exists a link ij such that

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right)\right] < 0 \iff$$
$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right)\right] > 0,$$

or equivalently,

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[b_{ij}'\left(x_{ij}^2 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c_i'\left(\sum_{p_j\in N_g(a_i)} x_{ij}^2\right)\right.$$
$$\left. - b_{ij}'\left(x_{ij}^1 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) + c_i'\left(\sum_{p_j\in N_g(a_i)} x_{ij}^1\right)\right] > 0,$$

contradicting $\boxed{\mathbf{A}}$. So, $\mathbf{x}_g^1 - \mathbf{x}_g^2 = \mathbf{0}$ and uniqueness is established. $\square$

Proof of Proposition 1. We now prove this by induction on N the number of agents. We can immediately check that a $2 = 2$ or a $3 = 3$ bipartite graph is a circular bipartite graph $\mathcal{C}_{2,2}$ or $\mathcal{C}_{3,3}$.

Assume for $N \geq 3$, that any $c = n$ bipartite graph with $c \leq N$ can be decomposed into circular bipartite graphs. Let g be a $c = n = N+1$ bipartite graph. Let $(a_{i_1}, p_{j_1}) \in A \times P$ with $i_1 j_1 \in L$. There exists $p_{j_2} \in P$ with $j_2 \neq j_1$ such that $i_1 j_2 \in L$ (since $r_{i_1}(g) = 2$) and then, there exists some $a_{i_2} \in A$ with $i_2 \neq i_1$ such that $i_2 j_2 \in L$ (since $r^{j_2}(g) = 2$).

If $i_2 j_1 \in L$, then g admits a $2 = 2$ bipartite subgraph and g', the restriction of g to $A\backslash\{a_{i_1}, a_{i_2}\}$ and $P\backslash\{p_{j_1}, p_{j_2}\}$, remains a $c = n = N-1$ bipartite graph, so by induction hypothesis g' allows a decomposition into circular bipartite graphs $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$. Thus, g is the disjoint union of $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$ and $\mathcal{C}_{2,2}$.

Otherwise, $i_2 j_1 \notin L$. So there exists $p_{j_3} \in P$ with $j_3 \neq j_1, j_2$ such that $i_2 j_3 \in L$. Then, there exists some $a_{i_3} \in A$ with $i_3 \neq i_1, i_2$ (since $r_{i_1}(g), r_{i_2}(g) \leq 2$) such that $i_3 j_3 \in L$. Again, if $i_3 j_1 \in L$, g admits a $3 = 3$ circular bipartite subgraph and g'', the restriction of g to $A \backslash \{a_{i_1}, a_{i_2}, a_{i_3}\}$ and $P \backslash \{p_{j_1}, p_{j_2}, p_{j_3}\}$, remains a $c = n = N - 2$ bipartite graph. Thus, g is the disjoint union of $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$ and $\mathcal{C}_{3,3}$.

Otherwise, $i_3 j_1 \notin L$, and so on. The process is finite because we may extract at most $N + 1$ public goods. In this case, the final link is $i_{N+1} j_1$. Hence, the $c = n = N + 1$ bipartite graph is precisely $\mathcal{C}_{N+1,N+1}$. □

Proof of Proposition 2. We may follow the same lines as in Hardy and Ramanujan (1918, Section 3, p. 88).

The number of unrestricted partitions of n is given by the coefficient of x^n in the expansion of the function,

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = 1 + \sum_{n=1}^{\infty} p(n)\, x^n,$$

for $|x| < 1$. Let g be defined by

$$\begin{aligned} g(x) &= \tfrac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots} \\ &= (1-x)\, f(x) \\ &= 1 + \textstyle\sum_{n=1}^{\infty} \left(p(n) - p(n-1)\right) x^n \\ &= 1 + \textstyle\sum_{n=2}^{\infty} p_2(n)\, x^n \\ &= \textstyle\sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Now, since the sequence $(a_n)_n \geq 0$ where $a_n = p_2(n)$ for $n \geq 2$ and $a_0 = 1$, $a_1 = 0$, and since $\ln g(x) \sim_1 \frac{\pi^2}{6}(1-x)^{-1}$, it follows by Theorem C in Hardy and Ramanujan (1917) that

$$\ln p(n) \sim \pi \sqrt{\frac{2n}{3}},$$

where $p(n) = 1 + \sum_{k=2}^{n} p_2(k) = \sum_{k=0}^{n} a_k$.

We can make the same reasoning for $p_2(n)$ as for $p(n)$. Let h be defined by

$$
\begin{aligned}
h(x) &= (1-x)\,g(x) \\
&= (1-x)\left(1+\textstyle\sum_{n=2}^{\infty} p_2(n)\,x^n\right) \\
&= \left(1+x+\textstyle\sum_{n=3}^{\infty} p_2(n)\,x^n\right) - \left(x+\textstyle\sum_{n=2}^{\infty} p_2(n)\,x^{n+1}\right) \\
&= 1+\textstyle\sum_{n=3}^{\infty} \left(p_2(n)-p_2(n-1)\right) x^n \\
&= \textstyle\sum_{n=0}^{\infty} b_n x^n,
\end{aligned}
$$

where $b_0 = 1$, $b_1 = b_2 = 0$ and $b_n = p_2(n) - p_2(n-1)$ for $n \geq 3$. Now, $(b_n)_n \geq 0$. Indeed any partition $(n_1, n_2, \ldots, n_K)$ of n with $n_1 \geq n_2 \geq \ldots \geq n_K$ into classes of size at least 2 can be *incremented* into $(n_1+1, n_2, \ldots, n_K)$, a partition of $n+1$ into classes at least 2. This mapping is one-to-one. Thus, $p_2(n) - p_2(n-1) \geq 0$. We have

$$(1-x)\ln h(x) = (1-x)\ln(1-x) + (1-x)\ln g(x) \to \frac{\pi^2}{6}$$

when $x \to 1$, since $(1-x)\ln(1-x) \to 0$ when $x \to 1$. Thus, $\ln h(x) \sim_1 \frac{\pi^2}{6}(1-x)^{-1}$. And then, by the second part of Theorem C in Hardy and Ramanujan (1917), it comes

$$\ln p_2(n) \sim \pi\sqrt{\frac{2n}{3}},$$

where $p_2(n) = p_2(2) + \sum_{k=3}^{n}(p_2(k) - p_2(k-1)) = \sum_{k=0}^{n} b_k$. $\square$

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