Stability in Matching with Couples: Maximal Domain and Paths*

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Abstract

We study matching with couples problems where hospitals have one vacant position. We introduce a property of couples' preferences over pairs of hospitals called *only-swap complementarity*. Our first result is that the domain of preferences satisfying *only-swap complementarity* is the only maximal domain for the existence of a stable matching that contains a natural class of preferences called *simple preferences*. We prove this result by showing that *only-swap complementarity* is equivalent to *weak substitutability* (Hatfield and Kojima, 2008) and *bilateral substitutability* (Hatfield and Kojima, 2010). We also extend Klaus and Klijn's (2007) paths to stability result by showing that if couples' preferences satisfy *only-swap complementarity*, then from any arbitrary matching there exists a finite path of matchings where each matching on the path is obtained by "satisfying" a blocking coalition for the previous one and the final matching is stable.

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1 Introduction

A matching with couples problem is a mathematical representation of a labor market with two salient features: (i) wages are fixed, and hence they cannot be used to equate labor supply and demand, and (ii) married couples participate in the market. An example of such labor market is the entry-level labor market for medical doctors in the U.S. which is administered by the National Resident Matching Program (NRMP).

Since the 1950s, the NRMP has used a variant of the Gale and Shapley (1962) algorithm to match doctors and hospitals. It was by 1970, that the increasing presence of married couples in the market led to a significant reduction of voluntary participation in the NRMP. This problem was tackled by allowing couples to express their preferences over pairs of hospitals.[1] The difficulties the NRMP experienced prior to its redesign suggest market outcomes were not "stable" in a way we describe next.

A blocking coalition consist of a group of doctors and hospitals that are not matched to each other but in fact would prefer to be. A matching is stable if there is no blocking coalition. In the presence of blocking coalitions the permanence of the matching is at serious risk as there are agents who have the incentive and the power to circumvent it.

Gale and Shapley (1962) demonstrate that in matching problems with no couples there is always a stable matching. Unfortunately, in the presence of couples the existence of a stable matching is no longer guaranteed (Roth, 1984).

The success of matching markets with couples such as the NRMP suggests that, despite the theoretical impossibility, stable matchings exist and are reached in real life applications.

In couples problems with a "small" number of agents, the existence of a stable matching can be guaranteed only under certain properties of couples preferences. Some such properties are: weak responsiveness (Klaus and Klijn, 2005 and Klaus et al., 2009), substitutability (Hatfield and Milgrom, 2005) and bilateral substitutability (Hatfield and Kojima, 2010).[2] By contrast, in couples problems where the number of agents is large the existence of a stable matching is guarantee as long as the proportion of couples is small

[1]See Roth and Peranson (1999) and Roth (2002) for details on the new design of the algorithm.

[2]Bilateral substitutability is weaker than both substitutability and weak responsiveness (Hatfield and Kojima, 2010).

relative to the number of hospitals (Kojima et al. 2013 and Ashlagi et al. 2014).[3,4]

We introduce a property of couples' preferences over pairs of hospitals called only-swap complementarity. Our property rules out most complementarities, but not all. We illustrate this point with the following example.

Suppose there is a couple formed by doctors Ana (A) and Ben (B). Each doctor receives several job offers from hospitals located in different cities. Consider two situations. Situation 1: A receives an offer from the hospital in Tijuana (Tij), and neither A nor B receive offers from the hospital in San Diego (SD). Situation 2: In addition to the offers A and B received in Situation 1, B receives an offer from the hospital in San Diego.[5]

We say there is a *complementarity* between Tij and SD, if A rejects the offer from Tij in Situation 1 and accepts it in Situation 2. Let h be the hospital chosen by B in Situation 1. If $h \neq Tij$ and couples' preferences satisfy only-swap complementarity, then A still rejects the offer from Tij in Situation 2. In this case there is no complementarity between Tij and SD. However, if $h = Tij$, then it can be that in Situation 2, A takes the job at Tij and B takes the job at SD. The swap of jobs between B and A motivates the name of our property.

Our first result is that the domain of preferences satisfying only-swap complementarity is the only maximal domain for the existence of a stable matching that includes a very narrow and natural preference domain: the domain of simple preferences. Another way to state our result is that, (i) if all couples' preferences satisfy only-swap complementarity, then there is a stable matching. And (ii) if one couple's preferences does not satisfy only-swap complementarity, then there are preferences for all hospitals, and simple preferences for one other couple such that regardless of the preferences of the other couples, no stable matching exists.[6]

[3]Kojima et al. (2013) and Ashlagi et al. (2014) consider random couples problems. They show that as the number of agents grows large the probability that a stable matching exists approaches one. For their results they require the growth rate of the number of couples to be small relative to the growth rate of the number of hospitals. In particular, Kojima et al. (2013) require that the growth rate of the number of couples is bounded by $\mathcal{O}(n^a)$, and Ashlagi et al. (2014) require the growth rate of the number of couples to be $n^{1-\epsilon}$, where n is the number of hospitals, $0 \leq a \leq \frac{1}{2}$ and $\epsilon > 0$.

[4]For an interdisciplinary and comprehensive review of the literature on couples problems see Biro and Klijn (2013).

[5]The job in Tijuana is not very desirable (location, work loads, reputation), whereas the job in San Diego is very desirable. Furthermore, Tijuana and San Diego are close to each other.

[6]For couples problems the domain of weakly responsive preferences is a maximal domain provided that couples' preferences satisfy an unemployment aversion property (see Klaus and Klijn (2005) and Klaus et al. (2009)). For the many-to-many matching with contracts problem, substitutability is a maximal domain (Hatfield and Kominers, 2012). The difference between the latter result and ours is that Hatfield and Kominers (2012) assume substitutability for all agents whereas we assume only-swap complementarity for couples. Since in the couples problem only-swap complementarity is strictly weaker

We show this result by proving that only-swap complementarity, bilateral substitutability and weak substitutability (Hatfield and Kojima, 2008) are equivalent in couples problems. This equivalence is important in its own as it makes clearer the relation between matching with couples and matching with contracts problems, where weak substitutability and bilateral substitutability were originally formulated.

Our second result is that for problems where all couples' preferences satisfy only-swap complementarity, we can reach a stable matching from any arbitrary matching by satisfying blocking coalitions one by one.[7] An implication of this result is that, starting from an arbitrary matching, certain random processes that match blocking coalitions, one at the time, converge to a stable matching with probability one.[8]

The importance of this result is that it provides theoretical support to the empirical observation that many decentralized matching markets perform well, suggesting they are able to reach stable outcomes (Kojima and Ünver, 2006).

Although we do not not explicitly consider single doctors, this is with no loss of generality as they can be easily incorporated in our framework and all our results would remain valid.

The remainder of the paper is organized as follows. In Section 2 we describe the matching with couples problem and introduce the only-swap complementarity property. In Section 3 we state the maximal domain and the paths to stability results. We conclude in Section 4. All proofs and the equivalence result are in the Appendices.

2 Matching with couples

There are two finite sets $\boldsymbol{H}$ and $\boldsymbol{C}$ of **hospitals** and **couples**. We denote generic elements of H and C by h and $c = (d_1, d_2)$, where d_1 and d_2 denote the spouses in couple c. Let $\boldsymbol{D} := \{d : d \in \{d_1, d_2\}$ for some $(d_1, d_2) \in C\}$ be the set of **doctors**. Each hospital has exactly one position to fill. Let $\boldsymbol{u}$ be the **outside option** for doctors. We can think of u

than substitutability their result does not imply ours. Obviously, our result does not imply theirs since the problem they consider is more general.

[7]Satisfying a blocking coalition: a couple or a hospital ends its partnership with unacceptable partners, or a couple and two hospitals match with each other, possibly in replacement of less preferred partners.

[8]This result is established for one-to-one matching problems by Roth and Vande Vate (1990). It is extended to many-to-many matching problems in which agents on one side have substitutable preferences and agents on the other side have responsive preferences by Kojima and Ünver (2006). For matching with couples problems, the paths to stability result holds whenever couples' preferences satisfy weak responsiveness (Klaus and Klijn, 2007).

as a hospital with no capacity constraint, so that each doctor can always find a job there.[9]

Each hospital $h \in H$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P_h}$ over the set D, and the prospect of having its position unfilled denoted by $\boldsymbol{\emptyset}$. For $d, d' \in D \cup \{\emptyset\}$, we write $d\, P_h\, d'$ if hospital h prefers d to d' $(d \neq d')$, and $d\, \boldsymbol{R_h}\, d'$ if h finds d at least as good as d', i.e., $d\, P_h\, d'$ or $d = d'$. If $d \in D$ is such that $d\, P_h\, \emptyset$, then d is an **acceptable doctor** for hospital h. By contrast, if $\emptyset\, P_h\, d,$ d is an unacceptable doctor for hospital h.

We represent hospitals' preferences by ordered lists of doctors and $\emptyset$; for example, $P_h = d_5, d_3, \emptyset \ldots$ indicates that hospital h prefers d_5 to d_3, and considers all other doctors to be unacceptable. Let $\boldsymbol{P_H} = \{P_h\}_{h \in H}$.

The restriction that each hospital has exactly one vacant position implies that no couple can get a job for each of its members in the same hospital. In other words, no pair (h, h) with $h \in H$ is **feasible**. The set of all feasible hospital pairs is given by

$$\bar{\boldsymbol{\mathcal{H}}} = [(H \cup \{u\}) \times (H \cup \{u\})] \setminus \{(h, h) : h \in H\}.$$

We denote a generic element of $\bar{\mathcal{H}}$ by (h, h').

Each couple $c = (d_1, d_2) \in C$ has a complete, transitive, and strict **preference relation** $\boldsymbol{P_c}$ over $\bar{\mathcal{H}}$. For each $(h_1, h_2), (h_3, h_4) \in \bar{\mathcal{H}}$ we write $(h_1, h_2)\, P_c\, (h_3, h_4)$ if c prefers d_1 and d_2 being matched to h_1 and h_2 respectively, to being matched to h_3 and h_4 respectively. We write $(h_1, h_2)\, \boldsymbol{R}_c\, (h_3, h_4)$ if c finds (h_1, h_2) at least as good as (h_3, h_4), i.e., $(h_1, h_2)\, P_c\, (h_3, h_4)$ or $(h_1, h_2) = (h_3, h_4)$. Any pair (h, h') such that $(h, h')\, R_c\, (u, u)$ is an **acceptable pair** to c and otherwise unacceptable.

We represent couples' preferences by means of ordered lists of feasible hospital pairs; for example, $P_c = (h_3, h_4), (h_5, h_3), (u, h_4), \ldots, (u, u) \ldots$ indicates that c prefers (h_3, h_4) to (h_5, h_3) and so on. Let $\boldsymbol{P_C} = \{P_c\}_{c \in C}$.

A one-to-one matching with couples problem or simply a **problem** is denoted by $\boldsymbol{(P_H, P_C)}$.

For each c we define a **choice function** Ch_c as

$$\boldsymbol{\mathsf{Ch}_c(\mathcal{H})} := \underset{P_c}{\text{argmax}} \big\{\mathcal{H} \cup \{(u, u)\}\big\}, \quad \text{for each } \ \mathcal{H} \subseteq \bar{\mathcal{H}}.$$

The choice function is defined on the set of feasible hospital pairs. Given a feasible set of hospital pairs, it selects the most preferred pair from the set and the outside option

[9]We refer to elements of $H \cup \{u\}$ as hospitals. When we refer only to elements in H we make it explicit by writing "hospitals in H".

(u, u). It is important to note that the choice function is not a primitive of our problem.

Let α be a property of a couple's preference relation. The set of all preferences satisfying α is called the **domain** of preferences satisfying α and is denoted by $\boldsymbol{\mathcal{P}_\alpha}$.

A **matching** specifies which hospitals are matched to which doctors. Formally, a matching $\boldsymbol{\mu}$ is a function defined on $D \cup H$ such that

- for each $d \in D$, $\mu(d) \in H \cup \{u\}$,
- for each $h \in H$, $\mu(h) \in D \cup \{\emptyset\}$,
- for each $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $\mu(h) = d$.

For each $c = (d_1, d_2) \in C$, we write $\boldsymbol{\mu(c)}$ to denote the pair $(\mu(d_1), \mu(d_2))$.

Now we introduce a central property of the matching literature: stability. Our stability concept is the same as the one in Klaus and Klijn (2005).

Let μ be a matching. A coalition $[\boldsymbol{h}]$ with $h \in H$ is a **blocking hospital** for μ if

- $\emptyset \, P_h \, \mu(h)$.

Let $c = (d_1, d_2) \in C$. A coalition $[\boldsymbol{c, (u, u)}]$, $[\boldsymbol{c, (\mu(d_1), u)}]$ or $[\boldsymbol{c, (u, \mu(d_2))}]$ is a **blocking couple** for μ if

- $\big(u, u\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$, $\big(\mu(d_1), u\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$ or $\big(u, \mu(d_2)\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$, respectively.

We often refer to blocking hospitals or to blocking couples as blocking coalitions.

A coalition $[\boldsymbol{c, (h, h')}]$ with $(h, h') \in \bar{\mathcal{H}}$ is a **blocking coalition** for μ if $(h, h') \notin \{(u, u), \big(\mu(d_1), u\big), \big(u, \mu(d_2)\big)\}$ and

- $\big(h, h'\big) \, P_c \, \big(\mu(d_1), \mu(d_2)\big)$;
- $[h \in H$ implies $d_1 \, R_h \, \mu(h)]$ and $[h' \in H$ implies $d_2 \, R_{h'} \, \mu(h')]$.

A matching is **stable** if there are no blocking coalitions. Since our analysis focuses on stability, whenever we specify a problem (P_H, P_C) it is enough to specify lists of acceptable doctors and lists of acceptable (and feasible) hospital pairs.

A set of hospital pairs is complete if (i) it contains the pair (u, u), and (ii) if combining the first and second components of any two pairs within the set results in a feasible hospital pair, then the latter pair also belongs to the set. Formally:

A subset $\mathcal{H} \subseteq \bar{\mathcal{H}}$ is **complete** if (i) $(u,u) \in \mathcal{H}$ and (ii) $\big[(h_1,h_2), (h_3,h_4) \in \mathcal{H}$ and $h_1 \neq h_4\big]$ imply $(h_1,h_4) \in \mathcal{H}$.

We define only-swap complementarity, which is a property of couples' preferences P_c over hospital pairs.

Only-swap complementarity, $\boldsymbol{P_c}$: for each complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$ and each h_1,h_2,h_3,h_4 such that $h_1,h_2 \notin \{u,h_3,h_4\}$, $(h_3,h_4) \in \mathcal{H}$, and $\mathsf{Ch}_c(\mathcal{H}) = (h_1,h_2)$, we have

$$\big[\textbf{(osc1)}\ (h_1,h_4)\, P_c\, (h_3,h_4) \quad \text{or} \quad \textbf{(osc2)}\ (h_1,u)\, P_c\, (h_3,h_4)\big]$$

and

$$\big[\textbf{(osc3)}\ (h_3,h_2)\, P_c\, (h_3,h_4) \quad \text{or} \quad \textbf{(osc4)}\ (u,h_2)\, P_c\, (h_3,h_4)\big].$$

Let $\boldsymbol{\mathcal{P}_{osc}}$ be the domain of preferences satisfying *only-swap complementarity.*

We explain *only-swap complementarity* by means of an example. Suppose married doctors Ana and Ben receive several offers from hospitals located in different cities. Suppose that they take the offer from the hospital in Tijuana (h_1) for Ana and the offer from the hospital in San Diego (h_2) for Ben.[10] Let $h_3 \neq h_1, h_2$ be a hospital that made an offer to Ana. Further, suppose that the offer from the hospital in San Diego is no longer available to Ben. In this case Ana rejecting the offer from Tijuana and taking the offer from h_3 while Ben taking an offer from a hospital $h_4 \neq h_1$ would be a violation of *only-swap complementarity.* However, Ana taking the offer from h_3 and Ben the offer from Tijuana ($h_4 = h_1$) is not a violation of *only-swap complementarity.* Complementarities that involve this kind of job swaps between couple's spouses are allowed by our property.

Now we introduce a natural preference domain: the domain of simple preferences. Simple preferences capture the situation in which a couple needs a job for only one of its members. In fact this is the way we incorporate single agents to our analysis i.e., by consider single agents as couples with simple preferences.

Simple preferences, $\boldsymbol{P_c}$: (i) $(h,h') \in \bar{\mathcal{H}}$ and $h' \in H \Longrightarrow (u,u)\, P_c\, (h,h')$, and (ii) there is a linear order $\succ$ on $H \cup \{u\}$ such that $h \succ h' \Longrightarrow (h,u)\, P_c\, (h',u)$.

Let $\boldsymbol{\mathcal{P}_{sim}} \subset \mathcal{P}_{osc}$ be the set of all simple preferences.

[10] Recall that the job in Tijuana is not very desirable (location, work loads, reputation), whereas the job in San Diego is very desirable. Moreover, Tijuana and San Diego are close to each other.

3 Results

Our first result is a maximal domain result. A preference domain $\mathcal{P}_\alpha$ is a **maximal domain** for the existence of a stable matching if the following holds:

- *Sufficiency.* If all couples' preferences are in $\mathcal{P}_\alpha$, then a stable matching exists.
- *Necessity.* Suppose there is at least two couples c and c'. If c's preferences are not in $\mathcal{P}_\alpha$, then there are preferences for hospitals and preferences for couple c' in $\mathcal{P}_\alpha$ such that regardless of the preferences of the other couples, no stable matching exists.

Theorem 1. *For problems with at least two couples, $\mathcal{P}_{osc}$ is the only maximal domain for the existence of a stable matching that contains $\mathcal{P}_{sim}$.*

Theorem 1 follows from Theorems A, B and Lemma 1 in Appendix A.

Hatfield and Kojima (2010) note that the couples problem can be obtained as a special case of the (matching with) contracts problem of Hatfield and Milgrom (2005) by making contracts to be such that, for each couple $c = (d, d')$ and each hospital h, there are two possible contracts between c and h: one that prescribes "d to match h" and another that prescribes "d' to match h". Each couple can sign at most two contracts (one for each member), and each hospital can sign at most one contract.[11] Moreover the concept of stable allocation of contracts coincides with our concept of stable matching.

We prove Theorem 1 by showing that *only-swap complementarity* is equivalent to one necessary and one sufficient condition for the existence of stable allocations in matching with contracts. It is possible to give a direct proof of Theorem 1. However, the equivalence between these conditions is interesting in its own as it makes more transparent the relation between couples and contracts problems.

3.1 Paths to stability

We extend Klaus and Klijn's (2007) path to stability result to the case where couples' preferences satisfy *only-swap complementarity.* We show that if all couples' preferences satisfy *only-swap complementarity*, there is always a path from an arbitrary matching to a stable one, such that each matching on the path is obtained by satisfying a blocking

[11] Note that a couple in the couples problem plays the role of a hospital in the contracts problem, and a hospital in the couples problem plays the role of a doctor in the contracts problem.

coalition for the previous matching. We first define precisely what we mean by "satisfying" a blocking coalition.

Satisfying blocking coalitions:[12] If $[h]$, $h \in H$ is a blocking hospital for a matching μ, then we say that a new matching ν is obtained from μ by satisfying the blocking coalition if h and $\mu(h)$ are unmatched, and all other agents are matched to the same mates at ν as they are at μ. Formally, matching ν is obtained from matching μ by satisfying blocking coalition $[h]$ for μ if

- $\nu(h) = \emptyset$ and $\nu(\mu(h)) = u$;
- $\nu(d) = \mu(d)$ for each $d \in D \setminus \{\mu(h)\}$;
- $\nu(\bar{h}) = \mu(\bar{h})$ for each $\bar{h} \in H \setminus \{h\}$.

Similarly, if $[c = (d_1, d_2), (h', h'')]$ is a blocking couple or a blocking coalition for a matching μ, then we say that a new matching ν is obtained from μ by satisfying the blocking coalition if (d_1, d_2) and (h, h') are matched to one another at ν, their mates at μ (if any, and if not involved in the blocking coalition) are unmatched at ν, and all other agents are matched to the same mates at ν as they were at μ. Formally, matching ν is obtained from matching μ by satisfying blocking coalition $[(d_1, d_2), (h', h'')]$ (for μ) if

- $\left[\mu(d_1) = h \in H \setminus \{h', h''\} \text{ implies } \nu(h) = \emptyset\right]$ and $\left[\mu(d_2) = h \in H \setminus \{h', h''\} \text{ implies } \nu(h) = \emptyset\right]$;
- $\left[\mu(h') = d \in D \setminus \{d_1, d_2\} \text{ implies } \nu(d) = u\right]$ and $\left[\mu(h'') = d \in D \setminus \{d_1, d_2\} \text{ implies } \nu(d) = u\right]$;
- $\nu(d_1) = h'$, $\nu(d_2) = h''$, $\left[h' \in H \text{ implies } \nu(h') = d_1\right]$, and $\left[h'' \in H \text{ implies } \nu(h'') = d_2\right]$;
- $\nu(d) = \mu(d)$ for each $d \in D \setminus \{\mu(h'), \mu(h''), d_1, d_2\}$;
- $\nu(h) = \mu(h)$ for each $h \in H \setminus \{\mu(d_1), \mu(d_2), h', h''\}$. $\Diamond$

Now we are ready to state our paths to stability result.

[12] We borrow the definition of satisfying blocking coalitions from Klaus and Klijn (2007, page 159).

Theorem 2 (*Paths to stability*). *Let (P_H, P_C) be a problem such that for each $c \in C$, $P_c \in \mathcal{P}_{osc}$. Let μ be an arbitrary matching for (P_H, P_C). Then, there is a finite sequence of matchings $\mu_1, \ldots, \mu_k$ such that $\mu_1 = \mu$, μ_k is stable, and for each $i = 1, \ldots, k-1$, there is a blocking coalition for μ_i such that μ_{i+1} is obtained from μ_i by satisfying this blocking coalition.*

The proof of Theorem 2 is relegated to Appendix B.

As a Corollary to Theorem 2 we obtain the following result. Consider a random process that begins by selecting an arbitrary matching μ and generates the sequence of matchings $\mu = \mu_1, \mu_2, \ldots$ where each μ_{i+1} is obtained from μ_i by satisfying a blocking coalition, chosen at random from the blocking coalitions for μ_i. Assume that the probability that any particular blocking coalition for μ_i is chosen to generate μ_{i+1} is positive, and only depends on the matching μ_i (but not on the number i). Let $\Psi(\mu)$ be the random sequence generated in this way from an initial matching μ.

Corollary 1 (*Random paths to stability*). *Let (P_H, P_C) be a problem such that for each $c \in C$, $P_c \in \mathcal{P}_{osc}$. For any initial matching μ for (P_H, P_C), the random sequence $\Psi(\mu)$ converges with probability one to a stable matching.*

To prove Theorem 2, we adapt the deterministic path algorithm to stability from Klaus and Klijn's (2007) DPC-Algorithm.[13] Our algorithm yields, in a finite number of steps, a stable matching for any problem in which couples' preferences satisfy *only-swap complementarity.*

In the description of our algorithm we use the aid of a virtual room that agents enter and exit throughout the algorithm. This visual devise was first introduced by Ma (1996) and is also used in Klaus and Klijn (2007).

3.2 Paths to Stability Algorithm (PS-algorithm)

Let μ be an arbitrary matching for a problem (P_H, P_C) where for each $c \in C$, $P_c \in \mathcal{P}_{osc}$.[14] After satisfying blocking hospitals for μ (first stage) we start putting couples one by one in an initially empty room (second stage). Each couple enters the room with its mates under μ. Whenever a couple enters the room with its mates, blocking coalitions within the

[13] The DPC-Algorithm of Klaus and Klijn (2007) is in turn a modification of the Roth and Vande Vate (1990) algorithm for one-to-one matching problems with no couples.

[14] This subsection follows closely Klaus and Klijn (2007, pages 161-163).

room are satisfied and the hospitals that are "dumped" are put outside the room. Thus, after this second stage we obtain a matching where all couples are matched to hospitals in the room, and for which there are no blocking coalitions within the room.[15] In the third stage, we let hospitals outside the room enter one by one. In each step possibly one blocking coalition within the room has to be satisfied before turning to the next step. The blocking coalitions that are satisfied in this stage are "hospital optimal" in the sense that for the hospital involved there is no other blocking coalition available within the room that would give it a better doctor. We call the doctor that is in all hospital optimal blocking coalitions associated with the entering hospital the *best doctor*. There may be several blocking coalitions that match the entering hospital with the best doctor. In order to assure the convergence of the algorithm we have to choose the blocking carefully. First, we prove (see the Claim in the third stage of the PS-algorithm and its proof in Appendix B) that one of the following is a blocking coalition: (a) the couple (to which the best doctor belongs), the hospital and the match of the best doctor's partner, (b) the couple, the hospital and the best doctor's match, or (c) the couple and the hospital. From these possible blocking coalitions we satisfy the blocking coalition that the couple prefers most. In the process of satisfying the blocking coalition at most two hospitals may exit the room.

We show that after a finite number of steps all hospitals have joined the couples in the room. Starting from μ we have obtained a stable matching for the problem (P_H, P_C). We now formalize the PS-algorithm.

A formal description of the PS-algorithm

Input: A problem (P_H, P_C) such that for each $c \in C$, $P_c \in \mathcal{P}_{osc}$, and a matching μ for (P_H, P_C).

Initialization: Set $A := \emptyset$. We call A the room.

- **First Stage**
  - Satisfy all blocking hospitals and blocking couples and denote the resulting matching by μ. `After Stage 1 we obtain a matching` $\mu^1 := \mu$ `with no blocking hospitals/couples.`
- **Second Stage**

[15] Up to stages 1 and 2 the PS-algorithm is exactly the same as the DPC-Algorithm of Klaus and Klijn (2007, pages 161-163). It is in the third stage where an adaptation is needed to deal with preferences that satisfy *only-swap complementarity* but do not satisfy weak responsiveness.

- If there is $c = (d_1, d_2) \in C \setminus A$, then let the couple and the hospitals in H assigned to it enter the room, i.e., set $A := (A \cup \{c, \mu(d_1), \mu(d_2)\}) \setminus \{u\}$.
- As long as there is a blocking coalition $[c' = (d'_1, d'_2), (h'_1, h'_2)]$ with $\{c', h'_1, h'_2\} \subseteq A \cup \{u\}$ do:

  Begin Loop: Satisfy $[c', (h'_1, h'_2)]$, and let dumped hospitals exit the room:

  * For $i = 1, 2$, [if $\mu(d'_i) = h \in H \setminus \{h'_1, h'_2\}$], then define $\mu(h) := \emptyset$ and set $A := A \setminus \{h\}$;
  * For $i = 1, 2$, if $h'_i \in H$ and $\mu(h'_i) = d \in D \setminus \{d'_1, d'_2\}$, then $\mu(d) := u$;
  * For $i = 1, 2$, define $\mu(d'_i) := h'_i$, and if $h'_i \in H$, then $\mu(h'_i) := d'_i$.

  End Loop

`After Stage 2 we obtain a matching` $\mu^2 := \mu$ `where all couples are in the room and there is no blocking coalitions.`

- **Third Stage**
  - As long as there is $h' \in H \setminus A$ do:

    Begin Loop: Set $A := A \cup \{h'\}$.

    If there is no blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$, then GO BACK to the beginning of the Third Stage. If there are blocking coalitions $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$, then let d'_1 be h''s most preferred doctor among the ones it could be matched to at these blocking coalitions. Let d'_2 be the partner of d'_1. Without loss of generality, $c' = (d'_1, d'_2) \in C$.
    Let $\boldsymbol{h_1^*} = \mu(d'_1)$, $\boldsymbol{h_2^*} = \mu(d'_2)$.

    Claim: $[c', (h', h_2^*)]$, $[c', (h', h_1^*)]$ or $[c', (h', u)]$ is a blocking coalition for μ.

    For each couple $c \in C$ and each matching ν, let

    $$\mathcal{B}(c, \nu) := \left\{(h_i, h_j) \in \bar{\mathcal{H}} : \ [c, (h_i, h_j)] \text{ is a blocking coalition for } \nu\right\}.^{16}$$

    Define

    $$(h', \hat{h}) = \mathsf{Ch}_{c'}\Big\{\{(h', h_2^*), (h', h_1^*), (h', u)\} \cap \mathcal{B}(c', \mu)\Big\}.$$

    The intersection above is non-empty by the Claim. Satisfy blocking coalition $[c', (h', \hat{h})]$, and if some hospitals are dumped (at most two), let them exit the

[16] This is the set of all hospital pairs that together with c form a blocking coalition for matching ν.

room. Formally,

define $\mu(c') := (h', \hat{h})$ and,

* $*$ **Case (a)** If $\hat{h} = h_2^*$ and $h_1^* \in H$, then define $\mu(h_1^*) = \emptyset$ and set $A := A \setminus \{h_1^*\}$.
* $*$ **Case (b)** If $\hat{h} = h_1^*$ and $h_2^* \in H$, then define $\mu(h_2^*) = \emptyset$ and set $A := A \setminus \{h_2^*\}$.
* $*$ **Case (c)** If $\hat{h} = u$, then for each $h^* \in \{h_1^*, h_2^*\} \cap H$, define $\mu(h^*) = \emptyset$ and set $A := A \setminus \{h_1^*, h_2^*\}$.

End Loop

`After Stage 3 we obtain a matching` $\mu^3 := \mu$ `where all couples and all hospitals are in the room and no blocking coalitions exist in the room.`

Output: A stable matching μ for (P_H, P_C).

Remark 1. One may wonder whether for any problem for which a stable matching exists, there exists some algorithm that starts in an arbitrary matching and converges to a stable one. The answer to this question is negative. This means that there are problems for which the set of stable matching is non-empty and no path of matchings obtained by satisfying blocking coalitions and starting from certain matching converges to a stable one. Example 4.1 of Klaus and Klijn (2007, page 167) exhibits a problem for which a stable matching exists and, starting from a certain matching, any path obtained by satisfying blocking coalitions cycles. As Klaus and Klijn (2007) point out: "this cycling has to do with the underlying complementarities in the couples' preferences, and not with the particular choice of the path (algorithm)."

Remark 2. The path to stability result generalizes to problems with couples and single doctors. We can incorporate single doctors by letting each single doctor have a fictitious partner that finds all hospitals unacceptable. For example, if single student d has preferences given by $P_d = h_1, h_2, u, h_4, \ldots$ then replace d by couple c with preferences $P_c = (h_1, u), (h_2, u), (u, u), (h_4, u) \ldots$ The path convergence result generalizes because the preferences of a fictitious couple induced by the preferences of a single doctor satisfy *only-swap complementarity*.

Example 1 (An application of the PS-algorithm)**.** Let (P_H, P_C) be the problem in Table 1. The sets of hospitals and couples are given by $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$ and $C = \{(d_1, d_2), (d_3, d_4), (d_5, d_6), (d_7, d_8), (d_9, d_{10})\}$. All couples' preferences satisfy *only-swap* *complementarity*. In Table 2 we apply the PS-algorithm to the initial matching $\mu(C) = (u,h_4),(h_5,u),(h_8,h_3),(h_2,h_1),(h_6,h_7)$. On the left hand side we give short explanatory comments that guide through the algorithm (we abbreviate the term blocking coalition by b.c.). On the right hand side we depict the matching at each point of the algorithm (whenever a hospital stands below a doctor it is matched to this doctor). In order to save space we abbreviate each couple (d_i,d_j) as d_id_j and each hospital pair (h_i,h_j) as h_ih_j. The vertical bar represents the door of the room: the agents on the left are inside the room and the agents on the right are outside the room. We obtain a path of matchings, each of them being the result of satisfying a blocking coalition for the previous matching. The output is the stable matching $\hat{\mu}(C) = (h_3,u),(u,h_1),(h_6,u),(h_5,h_2),(h_7,u)$.

We use Example 1 to explain two differences between the DPC-algorithm and the PS-algorithm. As can be seen in Table 2 the two algorithms coincide until step 11. In step 11 hospital h_3 enters the room and $[(d_1,d_2),(h_3,u)]$ is a blocking coalition. However, neither $[(d_1,d_2),(h_3,h_4)]$ nor $[(d_1,d_2),(h_3,h_1)]$ are blocking coalitions. In this situation the DPC-algorithm is not defined. By contrast, the PS-algorithm satisfies blocking coalition $[(d_1,d_2),(h_3,u)]$. In addition, note that at step 11 two hospitals are dumped.

Table 1: A problem where couples' preferences satisfy *only-swap complementarity*

P_C					P_H							
$P_{d_1d_2}$	$P_{d_3d_4}$	$P_{d_5d_6}$	$P_{d_7d_8}$	$P_{d_9d_{10}}$	P_{h_1}	P_{h_2}	P_{h_3}	P_{h_4}	P_{h_5}	P_{h_6}	P_{h_7}	P_{h_8}
h_3u	uh_1	h_6u	h_5h_2	h_6h_7	d_1	d_3	d_1	d_2	d_6	d_5	d_1	d_5
uh_2	h_2h_1	uh_5	h_5u	h_7h_6	d_4	d_1	d_2	d_1	d_8	d_9	d_5	
h_1h_4	h_2u	h_8u	uh_2	h_6u	d_3	d_2	d_3	d_3	d_7		d_9	
h_1u	h_5u		h_2u	h_7u	d_2	d_7	d_4	d_8	d_1			
uh_4				uh_7	d_5	d_8	d_6	d_5	d_2			
				uh_6				d_6	d_3			

Table 2: The PS-algorithm applied to (P_C, P_H)

Initial matching	$\vert d_1d_2$	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}
	$\vert uh_4$	h_5u	h_8h_3	h_2h_1	h_6h_7
(1) Stage 1					

Continued on next page

Table 2 – *Continued from previous page*

Satisfy one-sided blocking coalitions	$\vert d_1d_2$	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	h_1, h_3, h_7
	$\vert uh_4$	h_5u	h_8u	h_2u	h_6u	
(2) Stage 2						
choose couple (d_1d_2) to enter room	d_1d_2	$\vert d_3d_4$	d_5d_6	d_7d_8	d_9d_{10}	h_1, h_3, h_7
the room is stable	uh_4	$\vert h_5u$	h_8u	h_2u	h_6u	
(3) Stage 2						
choose couple (d_3d_4) to enter room	d_1d_2	d_3d_4	$\vert d_5d_6$	d_7d_8	d_9d_{10}	h_1, h_3, h_7
the room is stable	uh_4	h_5u	$\vert h_8u$	h_2u	h_6u	
(4) Stage 2						
choose couple (d_5d_6) to enter room	d_1d_2	d_3d_4	d_5d_6	$\vert d_7d_8$	d_9d_{10}	h_1, h_3, h_7
the room is unstable	uh_4	h_5u	h_8u	$\vert h_2u$	h_6u	
(5) Stage 2						
satisfy b.c. $[(d_5d_6), (uh_5)]$	d_1d_2	d_3d_4	d_5d_6	$\vert d_7d_8$	d_9d_{10}	h_1, h_3, h_7, h_8
the room becomes stable	uh_4	uu	uh_5	$\vert h_2u$	h_6u	
(6) Stage 2						
choose couple (d_7d_8) to enter room	d_1d_2	d_3d_4	d_5d_6	d_7d_8	$\vert d_9d_{10}$	h_1, h_3, h_7, h_8
the room is unstable	uh_4	uu	uh_5	h_2u	$\vert h_6u$	
(7) Stage 2						
satisfy b.c. $[(d_3d_4), (h_2u)]$	d_1d_2	d_3d_4	d_5d_6	d_7d_8	$\vert d_9d_{10}$	h_1, h_3, h_7, h_8
the room becomes stable	uh_4	h_2u	uh_5	uu	$\vert h_6u$	
(8) Stage 2						
choose couple (d_9d_{10}) to enter room	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\vert h_1, h_3, h_7, h_8$
the room is unstable	uh_4	h_2u	uh_5	uu	h_6u	
(9) Stage 2						
satisfy b.c. $[(d_5d_6), (h_6u)]$	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\vert h_1, h_3, h_5, h_7, h_8$
the room becomes stable	uh_4	h_2u	h_6u	uu	uu	
(10) Stage 3 [h_1 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\vert h_3, h_5, h_7, h_8$
satisfy b.c. $[(d_1d_2), (h_1h_4)]$	h_1h_4	h_2u	h_6u	uu	uu	
the room becomes stable						

Continued on next page

Table 2 – *Continued from previous page*

(11) Stage 3 [h_3 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\|h_1, h_4, h_5, h_7, h_8$
satisfy b.c. $[(d_1d_2), (h_3u)]$	h_3u	h_2u	h_6u	uu	uu	
(12) Stage 3 [h_1 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\|h_2, h_4, h_5, h_7, h_8$
satisfy b.c. $[(d_3d_4), (uh_1)]$	h_3u	uh_1	h_6u	uu	uu	
the room becomes stable						
(13) Stage 3 [h_2 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\|h_4, h_5, h_7, h_8$
satisfy b.c. $[(d_7d_8), (h_2u)]$	h_3u	uh_1	h_6u	h_2u	uu	
the room becomes stable						
(14) Stage 3 [h_5 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\|h_4, h_7, h_8$
satisfy b.c. $[(d_7d_8), (h_5h_2)]$	h_3u	uh_1	h_6u	h_5h_2	uu	
the room becomes stable						
(15) Stage 3 [h_7 enters the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	$\|h_4, h_8$
satisfy b.c. $[(d_9d_{10}), (h_7u)]$	h_3u	uh_1	h_6u	h_5h_2	h_7u	
the room becomes stable						
Stage 3 [h_4 and h_8 enter the room]	d_1d_2	d_3d_4	d_5d_6	d_7d_8	d_9d_{10}	
Final Output	h_3u	uh_1	h_6u	h_5h_2	h_7u	h_4, h_8

4 Conclusions

In this paper we study stability in matching with couples problems. Stability is important because stable matchings are robust to rematching by coalitions of agents. In this sense stable matchings are expected to last and are a good equilibrium prediction.

The presence of complementarities in couples' preferences may prevent the existence of a stable matching (Roth, 1984). As an example of complementarities in couples' preferences we can think of a couple of married doctors who wish to find jobs in the same city, and therefore the couple rejects or accepts a job for one of the spouses depending on whether it is possible to get a job for the other spouse in the same city.

Despite the theoretical impossibility, real world centralized and decentralized matching markets with couples seem to perform well, suggesting stable matchings are reached. We shed some light on this issue by studying which complementarities are compatible with (i) the existence of a stable matching, and (ii) the possibility of reaching a stable matching by means of a decentralized matching process.

More precisely, we show that (i) the domain of preferences satisfying *only-swap complementarity* is maximal for the existence of a stable matching, and (ii) if preferences satisfy *only-swap complementarity* then from any arbitrary matching there is a finite path of matchings such that each matching on the path is obtained by satisfying a blocking coalition from the previous one and the final matching is stable.

Appendix A

We define two properties of couples' preferences: bilateral substitutability and weak substitutability. Bilateral substitutability is sufficient for the existence of a stable matching in couples problems (Theorem A), while weak substitutability is necessary (Theorem B). In order to define these properties we first set up a matching with contracts version of the couples problem. Next, we show that bilateral substitutability and weak substitutability are equivalent to *only-swap complementarity* (Lemma 1). This result is interesting in its own as it helps to make more transparent the relation between matching with couples and matching with contracts problems.

Contracts

A **contract** is an ordered pair $\boldsymbol{(h,d)} \in H \times D$. The ordered pair $\boldsymbol{(u,d)}$ is a **null contract**. All the definitions in this section are made for couple $c = (d_1, d_2)$ and therefore **we drop all subindices and function arguments involving it** . The set of all possible contracts with members of couple c is $\boldsymbol{\bar{X}} := (H \cup \{u\}) \times \{d_1, d_2\}$. We denote the set of null contracts involving members of couple c by $\boldsymbol{U} := \big\{(u, d_1), (u, d_2)\big\}$.

The preference relation P over hospital pairs induces a **preference relation** $\tilde{P}$ **over sets of contracts**. Formally, for each (h_1, h_2), $(h_3, h_4) \in \bar{\mathcal{H}}$ we have

$$(h_1, h_2)\, P\, (h_3, h_4) \quad \text{if and only if} \quad \{(h_1, d_1), (h_2, d_2)\}\, \boldsymbol{\tilde{P}}\, \{(h_3, d_1), (h_4, d_2)\}.$$

We define a choice function $\boldsymbol{\tilde{\mathsf{Ch}}}$ as:

$$\boldsymbol{\tilde{\mathsf{Ch}}(X)} := \max_{\tilde{P}} \big\{\{(h, d_1), (h', d_2)\} \subseteq X \cup U \,:\, h, h' \in H \Longrightarrow h \neq h'\big\}, \text{ for each } X \subseteq \bar{X}.$$

We also define a rejection function as:

$$\boldsymbol{\tilde{\mathsf{Rej}}(X)} = X \setminus (\tilde{\mathsf{Ch}}(X) \cup U), \text{ for each } X \subseteq \bar{X}.$$

The rejection function gives for each $X \subseteq \bar{X}$, the set of contracts with hospitals in H that are rejected from X.

It can be easily verified that the choice function $\tilde{\mathsf{Ch}}$ satisfies **consistency** (Alkan, 2002):

$$\tilde{\mathsf{Ch}}(X'') \subseteq (X' \cup U) \subseteq (X'' \cup U) \quad \text{implies} \quad \tilde{\mathsf{Ch}}(X') = \tilde{\mathsf{Ch}}(X'').$$

The relation between Ch and $\tilde{\mathsf{Ch}}$ is stated in Claims 1 and 2 below, but first we need some additional definitions.

Let $\mathcal{H} \subseteq \bar{\mathcal{H}}$, we denote

- the sets of first and second components of the pairs in $\mathcal{H}$ by
  $\boldsymbol{H_1(\mathcal{H})} := \{h : (h,h') \in \mathcal{H}\}$ and $\boldsymbol{H_2(\mathcal{H})} := \{h' : (h,h') \in \mathcal{H}\}$,
- the set of contracts available to c when hospital pairs in $\mathcal{H}$ are available by
  $\boldsymbol{\bar{X}(\mathcal{H})} := \big(H_1(\mathcal{H}) \times \{d_1\}\big) \cup \big(H_2(\mathcal{H}) \times \{d_2\}\big)$.

Let $X \subseteq \bar{X}$, we denote

- the set of hospitals that have a contract with doctor $d \in \{d_1, d_2\}$ in X by
  $\boldsymbol{H(X,d)} := \{h \in H \cup \{u\} : (h,d) \in X\}$,
- the set of hospitals with contracts in X by
  $\boldsymbol{H(X)} := H(X,d_1) \cup H(X,d_2)$,
- the set of hospital pairs available to c when contracts in X are available by
  $\boldsymbol{\bar{\mathcal{H}}(X)} := \big\{(h,h') \in \bar{\mathcal{H}} \,:\, h \in H(X,d_1) \cup \{u\} \text{ and } h' \in H(X,d_2) \cup \{u\}\big\}$.

Claim 1. For each $X \subseteq \bar{X}$, $\tilde{\mathsf{Ch}}(X) = \{(h,d_1),(h',d_2)\} \implies \mathsf{Ch}(\bar{\mathcal{H}}(X)) = (h,h')$.

Claim 1 follows from the definitions of Ch, $\tilde{\mathsf{Ch}}$, and $\bar{\mathcal{H}}(\cdot)$.

Claim 2. For each complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$, $\mathsf{Ch}(\mathcal{H}) = (h,h') \implies \tilde{\mathsf{Ch}}(\bar{X}(\mathcal{H})) = \{(h,d_1),(h',d_2)\}$.

Claim 2 follows from the definition of a complete set of pairs, and the definitions of Ch, $\tilde{\mathsf{Ch}}$ and $\bar{X}(\cdot)$.

Bilateral substitutability is a property of each couple's choice function $\tilde{\mathsf{Ch}}$. For a couple, it means that if a job offer from hospital h to one of its members is rejected when all other available job offers come from different hospitals, the job offer is still rejected when a new job offer from a different hospital is received.

Bilateral substitutability (Hatfield and Kojima, 2010): there do not exist a set of contracts $X \subseteq \bar{X}$ and contracts $(h,d),(h',d') \in \bar{X}$ such that $h,h' \in H \setminus H(X)$, $(h,d) \notin \tilde{\mathsf{Ch}}\big(X \cup \{(h,d)\}\big)$ and $(h,d) \in \tilde{\mathsf{Ch}}\big(X \cup \{(h,d),(h',d')\}\big)$.

Theorem A (Sufficiency. Hatfield and Kojima, 2010)**.** *Let (P_H, P_C) be a problem. If all couples' preferences satisfy bilateral substitutability, then there is a stable matching for (P_H, P_C).*

Weak substitutability is a weakening of *bilateral substitutability* and it is not sufficient for the existence of a stable allocation in the contracts problem. Here we present its restriction to the couples problems. Intuitively, it means that the set of job offers rejected by the couple from a set of job offers, where no hospital offers a job to both members of the couple at the same time, expands when the couple receives new job offers from different hospitals.

Weak substitutability (Hatfield and Kojima, 2008): for each $X' \subseteq X'' \subseteq \bar{X}$ such that $\big[(h,d),(h',d') \in X''$ and $h = h' \in H\big]$ imply $[d = d']$, we have $\tilde{\mathsf{Rej}}(X') \subseteq \tilde{\mathsf{Rej}}(X'')$.

Theorem B (Necessity. Hatfield and Kojima, 2008)**.** *Suppose C contains at least two couples c and c'. Further suppose that P_c does not satisfy weak substitutability. Then, there exist $P_{c'} \in \mathcal{P}_{sim}$ and preferences for hospitals such that regardless of the preferences of the other couples, no stable matching exists.*

Lemma 1. *Bilateral substitutability, weak substitutability and only-swap complementarity are equivalent.*

Proof:

Weak substitutability $\Longrightarrow$ bilateral substitutability

Suppose *bilateral substitutability* does not hold. Then, there is a set of contracts $Y \subseteq \bar{X}$ and contracts (h,d), $(h',d') \in \bar{X}$ such that $h, h' \in H \setminus H(Y)$, (*) $\big[(h,d) \notin \tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})$ and $(h,d) \in \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})\big]$.[17]

Let $\boldsymbol{X'} = \big(\tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})\big) \cup \{(h,d)\}$ and $\boldsymbol{X''} = X' \cup \big\{(h',d')\big\}$.

Step 1: $\boldsymbol{(h,d) \notin \tilde{\mathsf{Ch}}(X')}$

We have that $\tilde{\mathsf{Ch}}(Y \cup \{(h,d)\}) \subseteq X' \cup U \subseteq (Y \cup \{(h,d)\}) \cup U$. Hence, by consistency, $\tilde{\mathsf{Ch}}(X') = \tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})$. Since $(h,d) \notin \tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})$, we conclude $(h,d) \notin \tilde{\mathsf{Ch}}(X')$.

Step 2: $\boldsymbol{\tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\}) = \{(h,d),(h',d')\}}$

By assumption $(h,d) \in \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})$. Assume that $(h,d) = (h',d')$ or $(h',d') \notin \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})$. Then, $\tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\}) \subseteq (Y \cup \{(h,d)\}) \cup U$. By consistency $\tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\}) = \tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})$. But, this is a contradiction to (*). So we have, $(h,d) \neq (h',d')$ and $(h',d') \in \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})$, which implies $\tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\}) = \{(h,d),(h',d')\}$.

[17] All statements are about couple $c = (d_1, d_2)$. Moreover, $d, d' \in \{d_1, d_2\}$.

Step 3: $(h, d) \in \tilde{\mathsf{Ch}}(X'')$.
It holds that

$$\tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\}) = \{(h,d),(h',d')\} \subseteq X'' \cup U \subseteq (Y \cup \{(h,d),(h',d')\}) \cup U.$$

By consistency, $\tilde{\mathsf{Ch}}(X'') = \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})$. Since $(h,d) \in \tilde{\mathsf{Ch}}(Y \cup \{(h,d),(h',d')\})$, we have $(h,d) \in \tilde{\mathsf{Ch}}(X'')$.

From steps 1 and 3 $\tilde{\mathsf{Rej}}(X') \nsubseteq \tilde{\mathsf{Rej}}(X'')$.

Step 4: X'' does not involve two different contracts with the same hospital in H.

This follows from (i) $h, h' \in H \setminus H(Y)$ and hence $h, h' \notin \tilde{\mathsf{Ch}}(Y \cup \{(h,d)\})$ and (ii) $\tilde{\mathsf{Ch}}$ never chooses the same hospital for both d_1 and d_2.

From $\tilde{\mathsf{Rej}}(X') \nsubseteq \tilde{\mathsf{Rej}}(X'')$ and Step 4 we conclude that *weak substitutability* does not hold. □

Bilateral substitutability $\Longrightarrow$ only-swap complementarity

Suppose *only-swap complementarity* does not hold. Then, there is a complete $\mathcal{H} \subseteq \bar{\mathcal{H}}$ and h_1, h_2, h_3, h_4 such that $h_1, h_2 \notin \{u, h_3, h_4\}$, $(h_3, h_4) \in \mathcal{H}$, $\mathsf{Ch}(\mathcal{H}) = (h_1, h_2)$, $(h_3, h_4)\, P\, (h_1, h_4)$ and $(h_3, h_4)\, P\, (h_1, u)$.[18] [19]

Let $\boldsymbol{X'} := \{(h_3, d_1), (h_4, d_2), (u, d_1), (u, d_2)\}$ and $\boldsymbol{X''} = \bar{X}(\mathcal{H})$.

Step 1: $(h_1, d_1) \notin \tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1)\})$.
Clearly, $(h_1, d_1), (h_2, d_2) \notin X'$. Moreover, the only pairs in $\bar{\mathcal{H}}(X' \cup \{(h_1, d_1)\})$ involving h_1 are (h_1, h_4) and (h_1, u). By assumption (h_3, h_4) is preferred to both of them. Hence, $(h_1, h_4), (h_1, u) \neq \mathsf{Ch}(\bar{\mathcal{H}}(X' \cup \{(h_1, d_1)\}))$. Therefore by Claim 1, $(h_1, d_1) \notin \tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1)\})$.

Step 2: $(h_1, d_1) \in \tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1), (h_2, d_2)\})$.
Claim 2 and $\mathsf{Ch}(\mathcal{H}) = (h_1, h_2)$ imply $\tilde{\mathsf{Ch}}(X'') = \{(h_1, d_1), (h_2, d_2)\}$. It also holds that $X' \subseteq X'' \cup U$. Therefore,

$$\tilde{\mathsf{Ch}}(X'') = \{(h_1, d_1), (h_2, d_2)\} \subseteq X' \cup \{(h_1, d_1), (h_2, d_2)\} \subseteq X'' \cup U.$$

[18] We are considering the case in which osc1 and osc2 fail. The case in which osc3 and osc4 fail is symmetric.
[19] Strict because $h_1 \neq h_3$.

Since $\tilde{\mathsf{Ch}}$ satisfies consistency,

$$\tilde{\mathsf{Ch}}(X' \cup \{(h_1, d_1), (h_2, d_2)\}) = \{(h_1, d_1), (h_2, d_2)\}.$$

Step 3: $\boldsymbol{h_1, h_2 \in H \setminus H(X')}$.
This follows from the definition of X' and the assumption that $h_1, h_2 \notin \{u, h_3, h_4\}$.

From Steps 1, 2 and 3, we conclude that *bilateral substitutability* does not hold. □

Only-swap complementarity $\Longrightarrow$ weak substitutability.
Suppose *weak substitutability* does not hold. Then there are sets $X' \subseteq X'' \subseteq \bar{X}$ such that (i) $\tilde{\mathsf{Rej}}(X') \nsubseteq \tilde{\mathsf{Rej}}(X'')$ and (ii) $[(h, d), (h', d') \in X''$ and $h = h' \in H$ imply $d = d']$.

From (i) there is $(h, d) \in \tilde{\mathsf{Rej}}(X') \setminus \tilde{\mathsf{Rej}}(X'')$. Without loss of generality let $d = d_1$. Let $h_1 = h$. Since $(h, d) \notin \tilde{\mathsf{Rej}}(X'')$ we have, (iii) $\tilde{\mathsf{Ch}}(X'') = \{(h_1, d_1), (h_2, d_2)\}$, for some $h_2 \in H \cup \{u\}$.

Let $\mathcal{H}' = \bar{\mathcal{H}}(X')$ and $\mathcal{H}'' = \bar{\mathcal{H}}(X'')$. By construction, $\mathcal{H}''$ is complete. Moreover, by Claim 1 and (iii) we have

$$\mathsf{Ch}(\mathcal{H}'') = (h_1, h_2) \ (\star).$$

Let $h_3, h_4 \in H \cup \{u\}$ be such that:

$$\mathsf{Ch}(\mathcal{H}') = (h_3, h_4) \quad (\star\star).$$

By construction, $(h_3, h_4) \in \mathcal{H}''$.

Step 1. $\boldsymbol{h_1 \notin \{u, h_3\}}$.
By $(\star\star)$ and Claim 2, $(h_3, d_1) \notin \tilde{\mathsf{Rej}}(X')$. Moreover, $(u, d_1) \notin \tilde{\mathsf{Rej}}(X')$ because $\tilde{\mathsf{Rej}}(X')$ only contains contracts with hospitals in H. Since $(h_1, d_1) \in \tilde{\mathsf{Rej}}(X')$, $h_1 \notin \{u, h_3\}$.

Step 2. $\boldsymbol{h_2 \notin \{u, h_4\}}$.
By Step 1, $(h_1, h_2) \neq (h_3, h_4)$. Suppose $h_2 \in \{u, h_4\}$. Then, as $(h_1, d_1) \in X'$ we have

$$\{(h_1, d_1), (h_2, d_2)\} \subseteq X' \cup U \subseteq X'' \cup U.$$

By consistency, $\tilde{\mathsf{Ch}}(X') = \{(h_1, d_1), (h_2, d_2)\}$, but this and Claim 1 contradict $\star\star$.

Step 3. $\boldsymbol{h_1 \neq h_4}$ and $\boldsymbol{h_2 \neq h_3}$.
It follows from $h_1, h_2 \in H$, $(h_1, d_1), (h_2, d_2), (h_3, d_1), (h_4, d_2) \in X''$ and (ii).

Step 4. $\neg$osc1 and $\neg$osc2 hold.
Since $(h_1, d_1), (h_3, d_1), (h_4, d_2) \in X'$, we have $(h_1, h_4), (h_3, h_4), (h_1, u) \in \mathcal{H}'$. Hence, relation $\star\star$ and $h_1 \neq h_3$ imply $(h_3, h_4)\, P\, (h_1, h_4)$ and $(h_3, h_4)\, P\, (h_1, u)$.

From steps 1 to 4, we conclude that P_c does not satisfy *only-swap complementarity* for $\mathcal{H}''$. □

Appendix B. The PS-algorithm is well defined

We prove Theorem 1 by showing that the PS-algorithm is well defined for problems where couples' preferences satisfy only-swap complementarity, i.e., we prove that given any such problem (P_H, P_C) and any matching μ for (P_H, P_C), the PS-algorithm produces a stable matching for (P_H, P_C) in a finite number of steps.[20]

Proof. We consider the three stages of the PS-algorithm. The first stage clearly is well defined and terminates in a finite number of steps. Also, the matching μ^1 at the end of the first stage does not have blocking hospitals.

The second stage also is well defined and terminates in a finite number of steps: there are only a finite number of couples and hence we only go through the loop a finite number of times. Moreover, the algorithm does not cycle in the loop since hospitals that are dumped are put outside of the room. For the matching μ^2 at the end of the second stage it holds that

- there are no blocking hospitals because (i) matching μ^1 does not have blocking hospitals and (ii) in the second stage all blocking coalitions that may be created in the room are removed by the loop.
- $C \subseteq A$ since the second stage terminates when all couples are in the room.
- for each $d \in D, \mu^2(d) \in A \cup \{u\}$, because (i) when a doctor is put in the room, the hospital he/she is matched to at that moment is put in the room as well and (ii) in the loop the hospitals that are not dumped remain in the room.
- there is no blocking coalition $[c', (h_1', h_2')]$ with $\{h_1', h_2'\} \subseteq A \cup \{u\}$ since in the loop these blocking coalitions are satisfied.

[20]The proofs in this Appendix follow closely those in Appendix A of Klaus and Klijn (2007).

We now proceed to prove that the third stage is well-defined, terminates in a finite number of steps, and that the output of the algorithm μ^3 is a stable matching.

We first prove that the third stage terminates in a finite number of steps. To this end we define a sequence which we prove to be strictly increasing in the number of loops of stage 3 and bounded from above.

For each $c \in C$ and each $(h, h') \in \bar{\mathcal{H}}$ let

$$\boldsymbol{r_c}\,(h, h') := \left|\{(h'', h''') \in \bar{\mathcal{H}} : (h'', h''')\, R_c\,(h, h')\}\right|$$

be the position of (h, h') in the preference list P_c. Denote by μ_k and n_k the matching and the number of hospitals in the room at the beginning of loop k, respectively.

Define the sequence $f_1, f_2, \ldots$ as:

$$f_k := \big(-2\sum_{c\in C} r_c\,(\mu_k(c))\big) + n_k, \quad k = 1, 2, \ldots$$

At each loop k of the third stage a hospital h' enters the room. Consider two cases.

Case 1. If there is no blocking coalition $[c', (h_1', h_2')]$ with $h' \in \{h_1', h_2'\}$. Then, the matching does not change, i.e., $\mu_{k+1} = \mu_k$. Therefore, $2\sum_{c\in C} r_c\,(\mu_{k+1}(c)) = 2\sum_{c\in C} r_c\,(\mu_k(c))$. At the same time the number of hospitals increases by one (since h' enters the room and no other hospital leaves it). Hence, $n_{k+1} = n_k + 1$. Hence, $f_{k+1} = f_k + 1$.

Case 2. If there is a blocking coalition $[c', (h_1', h_2')]$ with $h' \in \{h_1', h_2'\}$. Then, from the specific choice we make it follows that at the new matching μ_{k+1} one couple is strictly better off and no other couple changes mates. Hence, $-2\sum_{c\in C} r_c\,(\mu_{k+1}(c)) \geq -2\sum_{c\in C} r_c\,(\mu_k(c)) + 2$. At the same time, hospital h' entered the room and at most two hospitals (which were previously matched to members of the couple in the blocking coalition that was satisfied) exit the room. Therefore, $n_{k+1} \geq n_k - 1$. Summing up the two terms of f_{k+1} we conclude that $f_{k+1} \geq f_k + 1$.

Note that for all $k = 1, 2, \ldots$ the term $-2\sum_{c\in C} r_c\,(\mu_k(c))$ is bounded from above by $-2|C|$ and the term n_k is bounded from above by $|H|$. So, the sequence $f_1, f_2, \ldots$ is bounded from above by the number $-2|C| + |H|$.

The fact that the sequence $f_1, f_2, \ldots$ is strictly increasing and bounded from above implies that the third stage terminates in a finite number of steps.

It remains to show that the third stage is indeed well defined and that the final matching is stable. It suffices to show that the matching at the beginning of each loop satisfies the following properties:

(i) There is no blocking hospital or blocking couple;

(ii) There is no blocking coalition $[c', (h'_1, h'_2)]$ with $\{h'_1, h'_2\} \subseteq (A \cup \{u\}) \setminus \{h'\}$;

(iii) The Claim holds (which is conditional upon the existence of a blocking coalition $[c', (h'_1, h'_2)]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$).

Induction Basis: We prove that properties (i)-(iii) hold when the algorithm enters the loop of the third stage for the first time.

(i) and (ii): It follows from the properties of μ^2 that (i) and (ii) hold when the algorithm enters the loop of the third stage for the first time.

(iii): We prove that (iii) holds when the algorithm enters the loop for the first time. Assume that hospital $h' \in H \setminus A$ enters the loop, thus $A = A \cup \{h'\}$. Further, assume that there is a blocking coalition $\left[c', (h'_1, h'_2)\right]$ with $h' \in \{h'_1, h'_2\} \subseteq A \cup \{u\}$. Let d'_1 be h''s most preferred doctor among the ones it would get at these blocking coalitions. Let d'_2 be the partner of d'_1. Without loss of generality we assume $c' = (d'_1, d'_2) \in C$. Let $h^*_1 = \mu(d'_1)$ and $h^*_2 = \mu(d'_2)$.

Suppose to the contrary that (iii) does not hold. Then $\left[c', (h', h^*_2)\right]$, $\left[c', (h', h^*_1)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions. Hence, there is a blocking coalition $\left[c', (h', h'_3)\right]$ with $h'_3 \in A \setminus \{h^*_1, h^*_2\}$.
Consider the complete set of pairs $\mathcal{H}''$ depicted in the following table

(1) (h^*_1, h^*_2)	(4) (h', h^*_2)	(7) (u, h^*_2)
(2) (h^*_1, h'_3)	(5) (h', h'_3)	(8) (u, h'_3)
(3) (h^*_1, u)	(6) (h', u)	(9) (u, u)

First, we show that $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', h'_3)$. By (i), couple c' (weakly) prefers pair (1) to pairs (3), (7), and (9). By (ii), couple c' (strictly) prefers pair (1) to pairs (2) and (8). Since $\left[c', (h', h^*_2)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions, couple c' (strictly) prefers (1) to (4) and (6). Finally, since $\left[c', (h', h'_3)\right]$ is a blocking coalition, pair (5) is (strictly) preferred to (1) and therefore, by transitivity, (5) is preferred to all other pairs. This implies (a1) $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', h'_3)$.

By definition of $\mathcal{H}''$, $(\mathsf{a2})$ $(h_1^*, h_2^*) \in \mathcal{H}''$. Recall $h_3' \neq u$, h_1^*, h_2^*, and note that since h' just entered the room it must be that $h' \neq u$, h_1^*, h_2^*. Hence, $(\mathsf{a3})$ $h', h_3' \notin \{u, h_1^*, h_2^*\}$.

From $\mathsf{a1}$, $\mathsf{a2}$, $\mathsf{a3}$ and *only-swap complementarity* of P_c follows[21]

$$(h', h_2^*)\, P_{c'}\, (h_1^*, h_2^*) \quad \text{or} \quad (h', u)\, P_{c'}\, (h_1^*, h_2^*).$$

This contradicts the assumption that $\left[c', (h', h_2^*)\right]$ and $\left[c', (h', u)\right]$ are not blocking coalitions. We conclude (iii) holds.

Induction Assumption: Suppose that (i)-(iii) hold for loops 1 up to k of the third stage.

Induction Step: Now consider loop $k+1$ (where $k \geq 1$) of the third stage.

Since no agent is forced to accept an unacceptable agent in loop k, (i) is true. Using the arguments for (iii) of the first loop it is easy to check that (iii) is again true for loop $k+1$ if (ii) is also true for loop $k+1$. So, it only remains to prove that (ii) holds for loop $k+1$. It is clear that (ii) holds for loop $k+1$ if there is no blocking coalition $\left[c', (h_1', h_2')\right]$ with $h' \in \{h_1', h_2'\} \subseteq A \cup \{u\}$ for the matching at the end of loop k. We show that in fact this is the case.

Let μ_k and μ_{k+1} be the matchings at the beginning of loops k and $k+1$, respectively.[22] Assume that in loop k blocking coalition $[c', (h', \hat{h})]$ with $c' = (d_1, d_2)$ and

$$(h', \hat{h}) = \mathsf{Ch}_{c'}\Big(\{(h', \mu_k(d_2')), (h', \mu_k(d_1')), (h', u)\} \cap \mathcal{B}(c', \mu_k)\Big)$$

is satisfied. In the process of satisfying this blocking coalition, hospitals $\mu_k(d_1')$ and $\mu_k(d_2')$ may be dumped. Define h_a^*, h_b^* as follows,

$$h_a^* = \begin{cases} \mu_k(d_1') & \text{if } \mu_k(d_1') \text{ is dumped,} \\ u & \text{otherwise;} \end{cases}$$

$$h_b^* = \begin{cases} \mu_k(d_2') & \text{if } \mu_k(d_2') \text{ is dumped,} \\ u & \text{otherwise,} \end{cases}$$

then the agents in the room at the beginning of loop $k+1$ are $A \setminus \{h_a^*, h_b^*\}$.

To prove (ii) for loop $k+1$, we have to show that there is no blocking coalition $\left[\bar{c}, (\bar{h}, \tilde{h})\right]$ with $\{\bar{h}, \tilde{h}\} \subseteq (A \setminus \{h_a^*, h_b^*\}) \cup \{u\}$ for μ_{k+1}. Suppose, by contradiction, there is such a blocking coalition. Note that all agents remaining in the room [i.e., all agents in $(A \setminus \{h_a^*, h_b^*\})$] are (weakly) better off at μ_{k+1} compared to μ_k. So, $\left[\bar{c}, (\bar{h}, \tilde{h})\right]$ also blocks μ_k.

[21] Note (h', h_3', h_1^*, h_2^*) play the role of (h_1, h_2, h_3, h_4) in the definition of *only-swap complementarity*.

[22] Note that μ_{k+1} is also the matching at the end of loop k.

Hence, if $h' \notin \{\bar{h}, \tilde{h}\}$, then we obtain a contradiction to induction assumption (i) or (ii) for loop k. So, without loss of generality, $(\bar{h}, \tilde{h}) = (h', \tilde{h})$.

If $\bar{c} \neq c'$, then it follows immediately that in loop k hospital h' did not choose its optimal blocking mate; a contradiction. Similarly, if the blocking coalition in question equals $\left[\bar{c}, (\tilde{h}, h')\right]$, then $d'_2 \, P_{h'} d'_1$ and hospital h' did not choose its optimal blocking doctor; a contradiction. Hence, the blocking coalition we consider is of the form $\left[c', (h', \tilde{h})\right]$.

The table below depicts the complete set of pairs $\mathcal{H}''$.

(1) $(\mu_k(d'_1), \mu_k(d'_2))$	(4) $(h', \mu_k(d'_2))$	(7) $(u, \mu_k(d'_2))$
(2) $(\mu_k(d'_1), \tilde{h})$	(5) $(h', \tilde{h})$	(8) $(u, \tilde{h})$
(3) $(\mu_k(d'_1), u)$	(6) (h', u)	(9) (u, u)

Note that all pairs in $\mathcal{H}''$ are feasible. Obviously (1) is feasible. Moreover, the pairs (3), (6), (7), (8) and (9) are feasible because they contain u. Recall that $\mu_k(d'_1), \mu_k(d'_2), \tilde{h}$ are already in the room at loop k while h' enters the room at loop k. Therefore $h' \neq \tilde{h}$, $\mu_k(d'_1)$, $\mu_k(d'_2)$, and pairs (2), (4) and (5) are feasible.

Now, we show $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', \tilde{h})$. By induction hypothesis (i), couple c' (weakly) prefers pair (1) to pairs (3), (7) and (9). By induction hypothesis (ii), couple c' (strictly) prefers pair (1) to pairs (2) and (8). Now consider the blocking coalition that was satisfied, $\left[c', (h', \hat{h})\right]$. Since $\left[c', (h', \hat{h})\right]$ is a blocking coalition for μ_k, $(h', \hat{h})$ is preferred to (1). By definition, $(h', \hat{h})$ is (weakly) preferred to pairs (4) and (6). Summarizing, the pair $(h', \hat{h})$ is (weakly) preferred to pairs (1),(2),(3),(4),(6),(7),(8) and (9). Lastly, since $\left[c', (h', \tilde{h})\right]$ is a blocking coalition for μ_{k+1} and $\mu_{k+1}(c') = (h', \hat{h})$, $(h', \tilde{h}) \, P_{c'} \, (h', \hat{h})$. This implies (b1) $\mathsf{Ch}_{c'}(\mathcal{H}'') = (h', \tilde{h})$. Clearly, (b2) $(\mu_k(d'_1), \mu_k(d'_2)) \in \mathcal{H}''$.
We now show (b3) $h', \tilde{h} \notin \{u, \, \mu_k(d'_1), \, \mu_k(d'_2)\}$. Since h' enters the room in loop k, $h' \notin \{u, \, \mu_k(d'_1), \, \mu_k(d'_2)\}$. Since $\left[c', (h', \tilde{h})\right]$ is a blocking coalition for μ_{k+1}, $\left[c', (h', \tilde{h})\right]$ is also a blocking coalition for μ_k. Hence, by definition of $(h', \hat{h})$, if $\tilde{h} \in \{u, \, \mu_k(d'_1), \, \mu_k(d'_2)\}$, then $(h', \hat{h}) \, R_{c'} (h', \tilde{h})$. This contradicts $(h', \tilde{h}) \, P_{c'} \, (h', \hat{h})$.
By b1, b2, b3 and *only-swap complementarity* of $P_{c'}$, we conclude

$$(\mu_k(d'_1), \tilde{h}) \, P_{c'} \, (\mu_k(d'_1), \mu_k(d'_2)) \quad \text{or} \quad (u, \tilde{h}) \, P_{c'} \, (\mu_k(d'_1), \mu_k(d'_2)).$$

This contradicts inductive hypothesis (ii). □

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