EQUILIBRIUM FLOWS: BIPARTITE CONTRACTS IN LARGE MULTI-COMMODITY MARKETS

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Abstract. This article introduces a model of decentralized markets with frictions. In our framework, utility is imperfectly transferable between agents that can only trade through bipartite contracting. Economic outcomes are defined as pairs of a flow vector and a price vector.

We prove the existence of a competitive equilibrium outcome and discuss its efficiency. We interpret this equilibrium in the case of indivisible commodities. Under additional assumptions and in a partial equilibrium setting, we link this result with linear programming theories of network optimization and optimal assignment. We present two methods for the computation of such an equilibrium generalizing the simplex algorithm and ϵ-relaxation algorithm to extend those computation techniques to the imperfectly transferable utility settings.

As an illustration, we build a model for the overnight interbank loan market with counterparty risk, collateralization costs and risk aversion.

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1. Introduction

Models of pure exchange exchange economies usually make two key assumptions: the existence for all goods of a central exchange process in which agents can freely participate and that, on this market, agents face the same prices. However in many markets, initial producers can't directly reach - or if so at huge costs - final consumers. OTC financial markets, trade of agricultural goods in developing countries, illegal markets, artworks or diamonds are some examples that meet those characteristics. The frictions - geographic or social distance, dispersion of consumers, lack of information on the trustfulness of a trade partner - shape the market geometries and lead to the apparition of intermediaries (market makers such as wholesalers, transporters, distributors, brokers or retailers). Another consequence of these frictions is the existence of multiple prices for the same commodity depending on the agent location within the market.

In this paper we investigate the possibility of reaching an equilibrium outcome through bipartite contracting. The utility of agents is chosen to be quasi-linear in one commodity - the numeraire good, Shapley's utility money - and we assume this commodity to be imperfectly transferable. In order to map transfers between agents we introduce two vectors: a vector of flows and a vector of prices. A bipartite contract is described by a pair of agents, a flow of commodity and a price expressed in units of numeraire. We prove the existence of an equilibrium outcome in which no agent has interest to unilaterally deviate. We discuss the efficiency of the equilibrium and the case of indivisible goods. Under additional assumptions, we introduce two algorithms for the computation of an equilibrium outcome and highlight some of its properties.

As an example of the application of our theory, we propose a model for the overnight interbank loan market with counterparty risk, collateralization costs and risk aversion.

Relation to literature. It has been extensively pointed out in the classical economic literature that in real markets most actual transactions are bipartite. Matching models were introduced to give a micro-foundation to explain the equilibrium allocation of indivisible goods in two-sided markets. Built on these models, market design tools helped in designing efficient clearing houses to avoid the typical market failures (failure to provide thickness, congestion, make it safe to reveal its preference). Several settings have been described from Gale and Shapley (1962) for the non-transferable utility case and Shapley and Shubik (1971) for the transferable utility model to Galichon et al. (2014) that embodies both cases. This theory is closely related with the linear programming framework and computation methods of Dantzig (1951, 1963).

Contrary to the last articles that focus on one-to-one matching, Kelso and Crawford (1982) study the many-to-one case. In their settings, the transactions remain bilateral but agents may contract with several partners. They introduce the gross substitutes condition and show the existence of an equilibrium outcome for indivisible goods under this condition. Hatfield et al. (2013) generalized their result to a network of agents assuming that indivisible commodities were fully substitutable.

On the mathematical side, our work also relates to the results of Bertsekas (1986); Bertsekas and Eckstein (1986); Bertsekas (1998) that develop computational methods of resolution for min-cost flow problems on networks and linked these results with optimal assignment theory.

The main contribution of this article is the introduction of a framework generalizing those results by breaking the asymmetry of matching models and extending their application to a wide range of markets. To the best of our knowledge, our paper is the first to develop these tools.

Organization of the paper. In this article, we move from the most general framework to the most specific, adding assumptions when necessary. In section 2, we present our model of a multi-commodity market with bipartite contracting and imperfectly transferable utility. We illustrate the applicability of our settings through several examples. We show the existence of a competitive equilibrium and discuss its efficiency. In order to illustrate the difficulties met with indivisible commodities, we present in section 3 a one-to-many matching problem that does not verify the Kelso and Crawford (1982) gross substitutes condition. We discuss the relevance and interpretation of the equilibrium flows solution in this context. In section 4, under some additional hypotheses and in a partial equilibrium setting, we generalize two classical algorithms used in network optimization theory to compute an equilibrium outcome. These algorithms have simple economic interpretations. In section 5, we relate our results with the linear programming theory. It leads us to highlight several properties of the equilibrium from the cooperative game theory point of view. Finally, in section 6, as an example of the application of our theory, we include a model of the overnight interbank loan market with counterparty risk, collateralization costs and risk aversion.

2. Existence of equilibrium flows

2.1. **Settings of the model.** $\mathcal{I}$ is the set of the $N_{\mathcal{I}}$ agents that form our economy, $\mathcal{L}$ the set of the $N_{\mathcal{L}}$ traded commodities and m is money, our numeraire. Each agent $i \in \mathcal{I}$ has an initial endowment vector $e_i = [e_{i,l}]_{l \in \mathcal{L}}$ of commodities and a utility function $U_i(q_i, q_{i,m})$ associated with the consumption of the set of commodities $q_i = [q_{i,l}]_{l \in \mathcal{L}}$ and the wealth $q_{i,m}$. For each agent i, we assume that:

- **(A1):** U_i is continuous and derivable in 0.
- **(A2):** U_i is increasing.
- **(A3):** U_i is concave (convex preferences).
- **(A4):** U_i is quasilinear in money and each agent has a large initial wealth (deep pocket assumption).

The fourth assumption implies the existence of a monetary tool that agents may used to transfer utility between themselves. In Shapley's own words this means that utility is identified with money. The properties of quasilinear utility functions of the form $U_i(q_i, q_{i,m}) = V_i(q_i) + q_{i,m}$ are well known. In particular, a change in the initial endowment of money will not create a wealth effect and hence will not change the Marshallian demand as long as the initial wealth of all agents is large enough.

Those four first assumptions define consumptions preferences of agents. To describe their preferences over trade partners we use the imperfectly transferable utility setting, introduced by Galichon et al. (2014) for matching models. We define $\forall (ij, l) \in \mathcal{I}^2 \times \mathcal{L}$, the set of feasible marginal utility transfers $(p_{i,l}, p_{j,l}) \in \mathcal{F}_{ij,l}$ with $p_{j,l}$ the marginal quantity of numeraire sent by j and $p_{i,l}$ the marginal utility received by i.

- **(A5):** Sets of feasible marginal utility transfers $(\mathcal{F}_{ij,l})_{(ij,l) \in \mathcal{I}^2 \times \mathcal{L}}$ are:
  - Closed.
  - "South-East inclusive". If $(p_{i,l}, p_{j,l}) \in \mathcal{F}_{ij,l}$ and $p'_{i,l} \leq p_{i,l}$, $p'_{j,l} \geq p_{j,l}$ then $\left(p'_{i,l}, p'_{j,l}\right) \in \mathcal{F}_{ij,l}$.
  - "North-West bounded". If $p^n_{i,l} \underset{n \to +\infty}{\longrightarrow} +\infty$ and $p^n_{j,l}$ bounded above then $\exists N$ such that $\forall n \geq N$, $\left(p^n_{i,l}, p^n_{j,l}\right) \notin \mathcal{F}_{ij,l}$. Same property if $p^n_{i,l}$ is bounded below and $p^n_{j,l} \underset{n \to +\infty}{\longrightarrow} -\infty$.
  - Non-empty. If $p^n_{i,l} \underset{n \to +\infty}{\longrightarrow} -\infty$ and $p^n_{j,l} \underset{n \to +\infty}{\longrightarrow} +\infty$ then $\exists N$ such that $\forall n \geq N$ $\left(p^n_{i,l}, p^n_{j,l}\right) \in \mathcal{F}_{ij,l}$.

To describe the structure of a matching market with imperfectly transferable utility, Galichon et al. (2014) introduce $\forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L}$ a function $\mathcal{D}_{ij,l}$ - the distance to the bargaining set - positive inside the bargaining set, negative outside and null on the efficient border. In our setting it seems more appropriate to consider rent functions $(R_{ij,l})_{(ij,l)\in\mathcal{I}^2\times\mathcal{L}}$ such that $\forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L}$, $R_{ij,l}(p_{i,l}, p_{j,l}) = -\mathcal{D}_{ij,l}(u_{i,l}(p_{i,l}), -u_{j,l}(p_{j,l}))$, with $u_{i,l}$ and $u_{j,l}$ the marginal utility of each agent. $R_{ij,l}$ is proportional to the marginal utility generated by an infinitesimal increase in the transfer of commodity l between i and j. If the rent function is strictly negative the contract is not feasible, if it is strictly positive the contract is inefficient and if it is null the contract is both efficient and feasible.

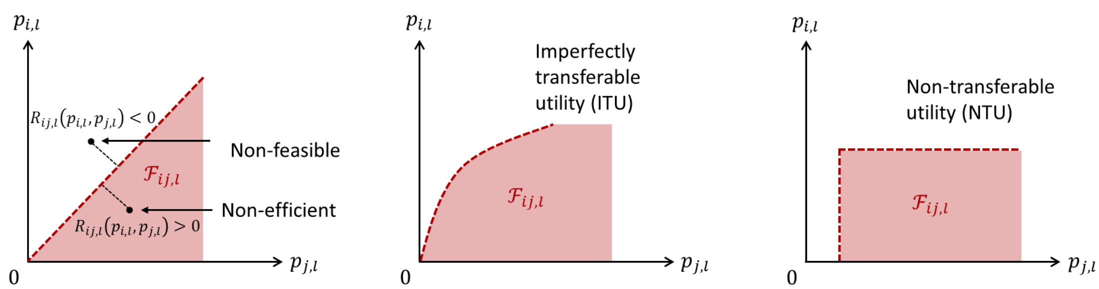

FIGURE 2.1. Bargaining sets and rent functions

(A5'): There exist rent functions $((p_{i,l}, p_{j,l}) \mapsto R_{ij,l}(p_{i,l}, p_{j,l}))_{(ij,l)\in\mathcal{I}^2\times\mathcal{L}}$ such that:
- $R_{ij,l}$ is continuous, positive inside $\mathcal{F}_{ij,l}$ and negative outside.
- $R_{ij,l}$ is increasing in $p_{j,l}$ and decreasing in $p_{i,l}$.
- $R_{ij,l}(p_{i,l} + a, p_{j,l} - a) = R_{ij,l}(p_{i,l}, p_{j,l}) - a$.

Galichon et al. (2014) prove (definition 1 and 2, lemma 1) that (A5) $\iff$ (A5').

Example. Imperfectly transferable utility settings:

Transportation cost. Assume there is a transportation cost $c_{ij,l}$ expressed in numeraire for the transportation of one unit of l between i and j. Then:

$$R_{ij,l}(p_{i,l}, p_{j,l}) = \frac{1}{2} \cdot (p_{j,l} - p_{i,l} - c_{ij,l})$$

Taxes. Consider there is an import duty or a tax on transactions $t_{ij,l}$ to be paid in numeraire for the transfer of one unit of l from i to j. Then:

$$R_{ij,l}(p_{i,l}, p_{j,l}) = \frac{1}{2 + t_{ij,l}} \cdot (p_{j,l} - p_{i,l} - t_{ij,l} \cdot p_{i,l})$$

Finally we make a last assumption in order to insure that the flow is bounded on cycles of traders:

(A6): $R_{ij,l}$ is such that there isn't any strictly profitable cycle.
Hence, for all cycle C there does not exist prices for a commodity l such that $\forall ij \in C,\ R_{ij,l}\left(p_{i,l}, p_{j,l}\right) \geq 0$ and $\exists ij \in C,\ R_{ij,l}\left(p_{i,l}, p_{j,l}\right) > 0$.

2.2. **Equilibrium outcomes.** A flow $\mu \in \left[\mu_{ij,l}\right]_{(ij,l)\in\mathcal{I}^2\times\mathcal{L}}$ is a mapping that gives us the transfer of commodities between agents in our economy. Thus, $\mu_{ij,l}$ is the quantity of commodity l transferred by agent i to agent j. In exchange for this transfer j pays $p_{j,l} \cdot \mu_{ij,l}$ and i receives $p_{i,l} \cdot \mu_{ij,l}$ units of the numeraire good.

Through these bipartite contracts, agent i bought or sold commodities. The operator "minus divergence" applied to the flow μ and computed for agent i - noted $\left(\nabla^*\mu\right)_i$ - is defined as the incoming flow minus the outgoing flow in i. It is the additional vector of commodities that i bought. We note:

$$\left(\nabla^*\mu\right)_i = \sum_h \mu_{hi} - \sum_j \mu_{ij}$$

$\left(\nabla^*\mu\right)_{i,l}$ is the l-th component of this vector. If $\left(\nabla^*\mu\right)_{i,l} > 0$, agent i bought more than he sold commodity l; if $\left(\nabla^*\mu\right)_{i,l} < 0$, agent i mostly sold this commodity. $\left(\nabla^*\mu\right)_{i,l} + e_{i,l}$ is the quantity of commodity l that agent i owns after trade happened. Hence $\left(\nabla^*\mu\right)_{i,l}$ is the balance of trade for commodity l and agent i and $\left(\nabla^*\mu\right)_i \cdot p_i$ is the balance of trade in monetary terms for agent i.

Definition. The set of Walrasian flows for a price vector p is:

$$\mathscr{W}\left(p\right) = \left\{\mu \in \mathbb{R}_+^{N_{\mathcal{I}}^2 \times N_{\mathcal{L}}} \text{ s.t. } \forall i \in \mathcal{I},\ \left(\nabla^*\mu\right)_i \in \arg\max_{q_i}\left[V_i\left(q_i + e_i\right) - q_i \cdot p_i\right]\right\}$$

As you may note, this definition implicitly contains the market clearing condition or feasibility condition. Indeed, by definition $\sum_{i\in\mathcal{I}} \left(\nabla^*\mu\right)_i = 0$. It implies that the total quantity of each commodity within the market remains the same regardless the set of contracts chosen by the agents.

In addition to satisfying this optimization problem for each agent, all contracts in the economy must be profitable (i.e. feasible): if $\mu_{ij,l} > 0,\ R_{ij,l}\left(p_{i,l}, p_{j,l}\right) \geq 0$. At equilibrium μ must also be unblocked: there does not remain an arbitrage (i.e. an inefficient contract) in our economy. So $\forall\left(ij, l\right) \in \mathcal{I}^2 \times \mathcal{L},\ R_{ij,l}\left(p_{i,l}, p_{j,l}\right) \leq 0$. If this last condition is not true, there exists a pair of agents that would both benefit from deviation by contracting between themselves.

Definition. The set of stable prices associated with a flow μ is:

$$\mathscr{S}(\mu) = \left\{ p \in \mathbb{R}_+^{N_\mathcal{I} \times N_\mathcal{L}} \text{ s.t. } \forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L}, \begin{array}{l} R_{ij,l}(p_{i,l}, p_{j,l}) \leq 0 \\ \text{if } \mu_{ij,l} > 0,\ R_{ij,l}(p_{i,l}, p_{j,l}) = 0 \end{array} \right\}$$

We note that a flow is stable if each agent solves the two discrete choices problem: sells to $j \in \arg\max_{j \in \mathcal{I}} [p_i \text{ s.t. } R_{ij}(p_i, p_j) = 0]$ and buys from $h \in \arg\min_{h \in \mathcal{I}} [p_i \text{ s.t. } R_{hi}(p_h, p_i) = 0]$. This means that agents are choosing the best trade opportunities they have.

At equilibrium the outcome (μ, p) must verify both conditions.

Definition. The set of equilibrium outcomes is:

$$\mathscr{E} = \left\{ (\mu, p) \text{ s.t. } \begin{array}{l} \mu \in \mathscr{W}(p) \\ p \in \mathscr{S}(\mu) \end{array} \right\}$$

2.3. **Existence and efficiency.** We prove the existence of an equilibrium using Kakutani's fixed point theorem. Although the proof does not give us a method of computation in the general case, it shows us the pertinence of our equilibrium.

Theorem. *If (A1)-(A6) are verified there exists an equilibrium outcome:*

$$\mathscr{E} \neq \emptyset$$

The following theorem is a reformulation of the welfare theorems in our settings. Although the results are the same, the conditions are different.

Theorem. *Welfare theorems:*

(1) $(\mu, p) \in \mathscr{E}$ *may not be Pareto efficient. However if* $\forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L}$ *the quantity* $[p_{i,l} - p_{j,l} - R_{ij,l}(p_{i,l}, p_{j,l})]$ *does not depend on prices then* $(\mu, p) \in \mathscr{E}$ *is Pareto efficient.*

(2) *If the initial endowment e is Pareto efficient and there is no subsidies for transaction -* $\forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L},\ p_{j,l} - p_{i,l} - R_{ij,l}(p_{i,l}, p_{j,l}) \geq 0$ *- then there exists a price vector p such that* $\left(\widetilde{0}, p\right) \in \mathscr{E}$ *(*$\widetilde{0}$ *is the flow null).*

If frictions are "distorted" and depend on the prices, for example if there are taxes or risk aversion, the equilibrium may not be efficient (see counter-example in appendix).

Apart from that case, an equilibrium is always efficient in the sense of Pareto. Similarly if there is no subsidy on our economy, the second welfare theorem is true and an efficient initial endowment is also an equilibrium.

3. Indivisibility and equilibrium flows

3.1. **The Kelso and Crawford (1982) condition.** On many markets, indivisibility is a key constraint for modeling. We start by going through a simple example of one-to-many matching that does not verify the gross substitutes assumption introduced by Kelso and Crawford (1982). Our purpose is to illustrate as clearly as possible the difficulties that one can meet with indivisible goods.

Consider the following classical problem of matching: two sellers $\{s_1, s_2\}$ can trade with two buyers $\{b_1, b_2\}$. Utility is perfectly transferable between one buyer and any seller; two sellers can't trade between themselves (frictions are large). Seller s_1 has an initial endowment of one unit of commodity l_1, and s_2 has one unit of l_2.

Utilities of consumption for buyers are:

$$\begin{cases} V_{b_1}\left(\{\emptyset\}\right) = 0 \\ V_{b_1}\left(\{l_1\}\right) = 0 \\ V_{b_1}\left(\{l_2\}\right) = 0 \\ V_{b_1}\left(\{l_1, l_2\}\right) = 3 \end{cases} \quad \text{and} \quad \begin{cases} V_{b_2}\left(\{\emptyset\}\right) = 0 \\ V_{b_2}\left(\{l_1\}\right) = 2 \\ V_{b_2}\left(\{l_2\}\right) = 2 \\ V_{b_2}\left(\{l_1, l_2\}\right) = 2 \end{cases}$$

and utility of consumption for sellers is null for any quantity of commodities, $V_{s_1} = V_{s_2} = 0$. Buyers transfer part of their utility of consumption to sellers through payment. In the case of a matching $(\{b_1; s_1\}; \{b_1; s_2\})$, agents will get utilities

$$\begin{cases} U_{b_1} = 3 - p_{b_1,l_1} - p_{b_1,l_2} \\ U_{b_2} = 0 \\ U_{s_1} = p_{b_1,l_1} \\ U_{s_2} = p_{b_1,l_2} \end{cases}$$

As said before this example typically does not verify the gross substitutes condition. Any matching is strictly dominated by $(\{b_1; s_1\}; \{b_1; s_2\})$ (b_1 consumes both commodities and the total utility generated is equal to 3). However at least one of the seller will earn less than 2 - which is what b_2 could pay him. So either $(\{b_1; s_1\}; \{b_1; s_2\}) \preceq$

$(\{b_1; s_1\}; \{b_2; s_2\})$ or $(\{b_1; s_1\}; \{b_1; s_2\}) \preceq (\{b_2; s_1\}; \{b_1; s_2\})$. Hence all matchings are blocked and the problem does not have a stable solution.

3.2. **The flow solution.** Let's now assume that commodities are divisible and that utility functions are continuous in the quantities consumed. We proved in the last section that there exists a competitive equilibrium set of contracts.

As an example we choose the following continuous and concave utility functions for consumption that verify the discrete conditions of the matching example in the last paragraph:

$$\begin{cases} V_{b_1}\left(\begin{bmatrix} \mu_{s_1 b_1, l_1} \\ \mu_{s_2 b_1, l_2} \end{bmatrix}\right) = \min\left(3\mu_{s_1 b_1, l_1}; 3\mu_{s_2 b_1, l_2}\right) \\ V_{b_2}\left(\begin{bmatrix} \mu_{s_1 b_2, l_1} \\ \mu_{s_2 b_2, l_2} \end{bmatrix}\right) = \min\left(2\mu_{s_1 b_2, l_1} + 2\mu_{s_2 b_2, l_2}; 2\right) \end{cases}$$

The output (μ, p) such that

$$\mu = \begin{bmatrix} \mu_{s_1 b_1, l_1} \\ \mu_{s_2 b_1, l_2} \\ \mu_{s_1 b_2, l_1} \\ \mu_{s_2 b_2, l_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } p = \begin{bmatrix} p_{b_1, l_1} \\ p_{b_1, l_2} \\ p_{b_2, l_1} \\ p_{b_2, l_2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

is unblocked as neither sellers nor buyers can improve their situation by changing unilaterally their strategy.

Hence after transfers, agents receive the utility $U_{s_1} = U_{s_2} = \frac{3}{2}$, $U_{b_1} = 0$ and $U_{b_2} = \frac{1}{2}$. There exists a stable but non-integer flow. Nevertheless several arguments allow us to apply the flow solution to markets with indivisible commodities such as housing market. First, following Azevedo et al. (2013), we could argue that the market is large and that flows are proportion of sellers of one type matched with each type of buyers. We could also interpret flows as the output of mixed strategies of sellers and buyers.

4. Computation of a partial equilibrium

4.1. **Additional assumptions.** The theorems in section 2 aim at demonstrating the relevance of our definition of an equilibrium outcome. However these results don't allow for the computation of an equilibrium and the estimation of the parameters of the model. In the following section we add assumptions to our model and introduce two computation methods for an equilibrium outcome. These additional assumptions obviously limit the scope of applications for the model. However we consider that this setting gives a good intuition of the mechanisms that lead to the apparition of flows and the formation of prices in a decentralized market cleared by bilateral contracting.

All agents $i \in \mathcal{I}$ in our economy have an initial endowment $e_i \in \mathbb{R}_+$ of one commodity and a large allocation of imperfectly transferable numeraire. As we only consider one commodity, $\forall ij \in \mathcal{I}^2$ and $\forall i \in \mathcal{I}$, μ_{ij} and p_i are reals (not vectors).

For each agent $i \in \mathcal{I}$ we consider that their utility functions and excess demand correspondences are described by the graphs on figure 4.1. Any agent described by the general excess demand function $z_i(p_i)$ (top right of figure 4.1) is a linear combination of the three agents described on the bottom. We can restrict ourselves to consider only those three without any loss of generality.

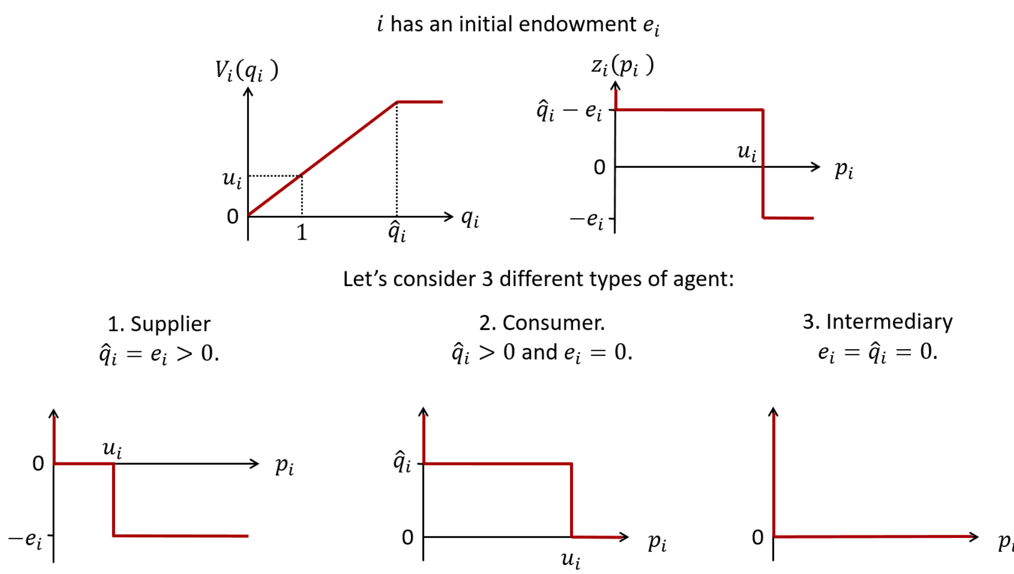

Figure 4.1. Utility, initial endowments and excess demand functions

These supply and demand functions are typically used in matching models. Section 6 justifies this assumption precisely in the case of the overnight interbank loan market.

In the general case, we can first interpolate any quasi linear utility function by a piecewise linear function; then express the resulting excess demand function as a sum of several inflexible excess demand functions. Therefore results on the computation of equilibrium for agents with utility functions described in figure 4.1 can be applied to approximate any partial equilibrium for agents with quasi linear preferences.

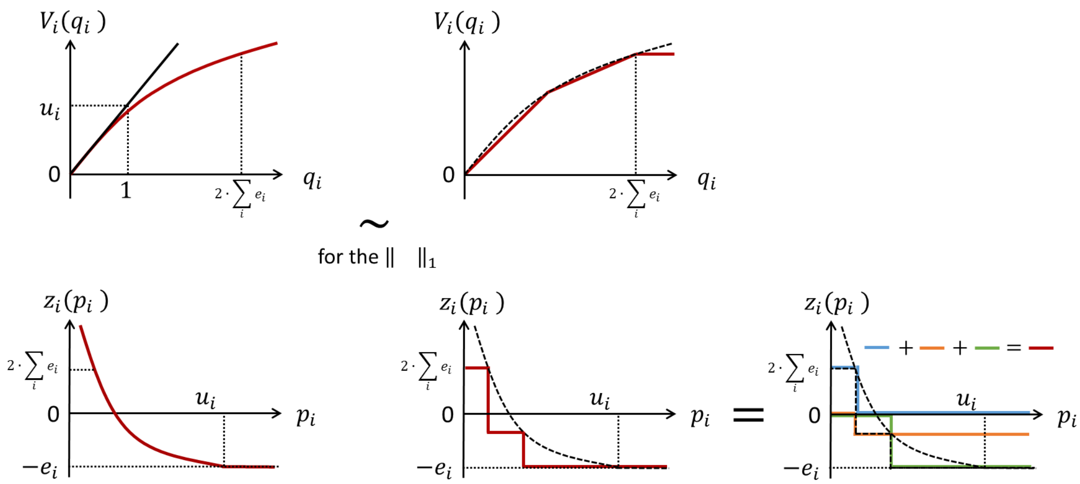

FIGURE 4.2. Linear interpolation of quasi linear utility functions

4.2. **Equivalent formulation for the consumer's problem.** Under the above assumptions, the consumer's optimization problem can be expressed as a discrete choice problem. To formalize this idea, we add a fictive agent, the "reservation player" noted 0 and a set $\mathcal{I}'$ of "slack agents", $\forall i \in \mathcal{I}$:

(1) $u_0 = 0$, $p_0 = 0$ and $z_0 = -\sum_{i\in\mathcal{I}} z_i$.
(2) $z_i = 0$.
(3) If i is a supplier ($z_i < 0$) we add a slack player $i' \in \mathcal{I}'$ such that $z_{i'} = -e_i$, $R_{i'i}(p_{i'}, p_i) = \frac{1}{2}\cdot(p_i - p_{i'})$ and $R_{i'0}(p_{i'}, p_0) = \frac{1}{2}\cdot(p_0 - p_{i'} + u_i)$.
(4) If i is a consumer ($z_i > 0$) we add a slack player $i' \in \mathcal{I}'$ such that $z_{i'} = q_i$, $R_{ii'}(p_i, p_{i'}) = \frac{1}{2}\cdot(p_{i'} - p_i)$ and $R_{0i'}(p_0, p_{i'}) = \frac{1}{2}\cdot(p_{i'} - p_0 - u_i)$.

We define $\mathcal{I}'_0 = \mathcal{I} \cup \mathcal{I}' \cup \{0\}$. If an agent chooses to trade with the reservation player 0, she is choosing her reservation choice. The set of slack players is needed to be sure that we are not adding a negative cost cycle during this transformation.

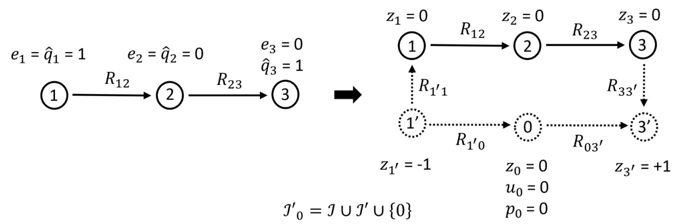

FIGURE 4.3. Example - Economy transformation

We now have a fixed supply or demand $\forall i \in \mathcal{I}_0',\ z_i(p) = z_i$ without any loops of negative costs in our new economy. Moreover the problem remains the same: for example player 1 has been divided in two players 1 and $1'$ that are choosing between their reservation choice (agent 0) and the trading opportunities they have in the economy.

Then, the set of Walrasian flows does not depend anymore on prices

$$\mathscr{W}_0 = \left\{\mu \in \mathbb{R}_+^{(2\cdot N_{\mathcal{I}}+1)^2} \text{ s.t. } \forall i \in \mathcal{I}_0',\ (\nabla^* \mu)_i = z_i\right\}$$

and the set of equilibrium outcomes for the set of agents $\mathcal{I}_0'$ is

$$\mathscr{E} = \left\{(\mu, p) \text{ s.t. } \begin{array}{l} \mu \in \mathscr{W}_0 \\ p \in \mathscr{S}(\mu) \end{array}\right\}$$

The next paragraphs presents two algorithms that can be used to compute an equilibrium outcome.

4.3. **Generalized simplex algorithm.** This algorithm is based on the simplex algorithm introduced by Dantzig (1963, 1951). All steps of the mechanism are given in appendix. In this section we focus on its principle and its economic interpretation.

- We add an extremely expensive trade partner for all suppliers and consumers to clear the market.
- At initialization all agents choose to trade with this agent.
- From then on, agents exploit arbitrages sequentially, price being driven by the most competitive trade opportunity.

Theorem. *If there does not exist a cycle of negative cost on* $G_0 = \left(\mathcal{I}_0', (\mathcal{I}_0')^2\right)$ *then:*

(1) *This mechanism ends after a finite number of iterations.*
(2) *It returns an equilibrium outcome* $(\mu, p) \in \mathscr{E}$ *such that if* $\forall i \in \mathcal{I}_0',\ z_i \in \mathbb{Z}$ *then* $\mu \in \mathbb{N}^{2\cdot N_{\mathcal{I}}+1}$ *(commodity can be indivisible).*

4.4. **Decentralized auction algorithm.** The mechanism is adapted from the ϵ-relaxation method introduced by Bertsekas (1986). It has a natural economic interpretation:

- Agents are competing in an auction to reach their needs of commodity. Traders observe prices for agents that are their immediate neighbors.
- The auction has a minimum bidding rule.
- Each agent is choosing the most profitable deal he can have.
- Agents with an excess of commodity decrease their prices while those with a lack of commodity increase their prices.

Theorem. *If there does not exist a cycle of negative cost then:*

(1) *This mechanism converges toward a limit outcome after a finite number of iterations.*
(2) *It returns an ϵ-equilibrium outcome.*

4.5. **Bargaining set and price discrimination.** In the last paragraphs, agents can't extract a strictly positive surplus from being a pure intermediary. This makes sense as long as a consumption vector can't be solution of an agent's optimization problem for two different price vectors; hence as long as a rational consumer can't sustain price discrimination. Under the additional assumptions of section 4, initial owners and final consumers are sharing the total surplus in a stable way (in the sense of matching models). We now allow intermediaries to discriminate their trade partners over prices. This won't change the equilibrium flows we define, but will reallocate the numeraire good at equilibrium.

For each transaction $ij \in (\mathcal{I}'_0)^2$, i sells at price $p_{i,ij}$ and j buys at $p_{j,ij}$. The general definition of Walrasian flows becomes:

$$\mathscr{W}(p) = \left\{ \mu \in \mathbb{R}_+^{N_{\mathcal{I}}^2 \times N_{\mathcal{L}}} \text{ s.t. } \forall i \in \mathcal{I},\ (\nabla^* \mu)_i \in \bigcap_{j \neq i} \arg\max_{q_i} \left[V_i(q_i + e_i) - q_i \cdot p_{i,ij} \right] \right\}$$

We introduce the notations

$$\begin{cases} \forall i \in \mathcal{I}'_0,\ (\nabla^* \mu p)_i = \left(\sum\limits_{ij \in (\mathcal{I}'_0)^2} \mu_{ij} \cdot p_{i,ij} \right) - \left(\sum\limits_{hi \in (\mathcal{I}'_0)^2} \mu_{hi} \cdot p_{i,hi} \right) \\ \forall ij \in (\mathcal{I}'_0)^2,\ (\nabla p)_{ij} = p_{j,ij} - p_{i,ij} \end{cases}$$

$(\nabla^* \mu p)_i$ is the quantity of utility "intercepted" by an intermediary, the benefits the agent is making by buying some commodity at a price and selling it at a higher price. $(\nabla p)_{ij}$ is the marginal quantity of numeraire lost because of the frictions in the transfer of utility. It can be interpreted as a cost and its value is set by the condition $R_{ij}(p_{i,ij}, p_{j,ij}) = 0$.

For a price vector p such that $\forall ij \in (\mathcal{I}'_0)^2$ $R_{ij}(p_{i,ij}, p_{j,ij}) = 0$, a given flow $\mu \in \mathscr{W}_0$ generates the total surplus

$$V_\mu(p) \quad = \sum_{i \in \mathcal{I}'_0} u_i \cdot z_i - \sum_{ij \in (\mathcal{I}'_0)^2} \mu_{ij} \cdot (\nabla p)_{ij}$$

and for each agent the participation constraint is $\forall i \in \mathcal{I}'_0,\ (\nabla \mu p)_i \geq 0$ and $\forall i \in \mathcal{I}' \cup \{0\}$, $(\nabla^* \mu p)_i = 0$ so equilibrium allocations of numeraire verifies:

$$\begin{cases} \sum_{i \in \mathcal{I}'_0} (\nabla^* \mu p)_i = V_\mu(p) \\ \forall i \in \mathcal{I},\ (\nabla^* \mu p)_i \geq 0 \\ \forall i \in \mathcal{I}' \cup \{0\},\ (\nabla^* \mu p)_i = 0 \end{cases}$$

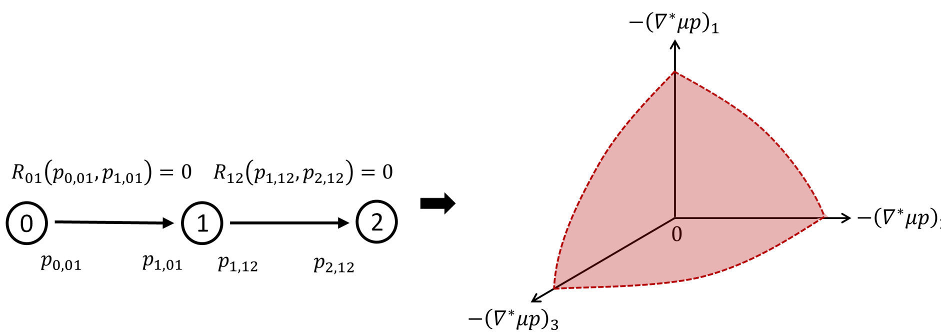

Figure 4.4. Example of bargaining set

5. Links with linear programming and cooperative game theory

The results presented in this article include as a particular case some well-known conclusions of linear programming and cooperative game theory. We will more specifically relate our finding with theory of matching and network optimization.

In this section we assume that $\forall ij \in \mathcal{I}^2$, $p_j - p_i - R_{ij}(p_i, p_j) = c_{ij}$ is a constant (it does not depend on the price).

5.1. **Characteristic function of the cooperative game.** The characteristic function of a cooperative game gives the utility generated by a coalition. It is a practical way of studying a cooperative game.

For all coalitions $S \subseteq \mathcal{I}$, we define $S_0' = S \cup S' \cup \{0\} \subseteq \mathcal{I}_0'$ using the transformation presented in section 4.2 to add a reservation player and a set of slack players.

Proposition. *The characteristic function of the game is*

$$v(S) = \max_{\mu_{|S_0'} \in \mathscr{W}_{S_0'}} \left[\sum_{i \in S_0'} u_i \cdot z_i - \sum_{ij \in \left(S_0'\right)^2} \mu_{ij} \cdot c_{ij} \right]$$

5.2. **Core.** By definition the core is the set of allocations that can not be improved by a subset of agents. And we know that for all coalitions $S \subseteq \mathcal{I}$:

$$\begin{aligned} v(S) &= \max_{\mu_{|S_0'} \in \mathscr{W}_{S_0'}} \left[\sum_{i \in S_0'} u_i \cdot z_i - \sum_{ij \in \left(S_0'\right)^2} \mu_{ij} \cdot c_{ij} \right] \\ &= \sum_{i \in S_0'} u_i \cdot z_i - \min_{\mu_{|S_0'} \in \mathscr{W}_{S_0}} \left[\sum_{ij \in \left(S_0'\right)^2} \mu_{ij} \cdot c_{ij} \right] \end{aligned}$$

Proposition. *The core of the trade game with constant marginal costs is the set of min-cost flows:*

$$\mathscr{C} = \left\{ \mu \ s.t. \ \mu = \underset{\mu \in \mathscr{W}_0}{argmin} \left[\sum_{ij \in \left(\mathcal{I}_0'\right)^2} \mu_{ij} \cdot c_{ij} \right] \right\} \neq \emptyset$$

Remark. We can compute an outcome in the core using the simplex algorithm, Dantzig (1963).

Proposition. *The duality of the min-cost flow problem gives:*

$$\mu \in \mathscr{C} \Leftrightarrow \exists p \ s.t. \ (\mu, p) \in \mathscr{E}$$

We also know that:

1. *Equilibrium outcomes are efficient.*
2. *Commodity can be indivisible.*

Remark. By definition the core is not a notion that embodies the idea of collusion. This is not a framework for bargaining on a network.

5.3. **Bargaining set and price discrimination.** We use the notations and the example of paragraph 4.3 to describe the stable allocations of numeraire. A flow $\mu \in \mathscr{W}_0$ generates the surplus

$$V_\mu = \sum_{i \in \mathcal{I}'_0 \text{ s.t. } z_i > 0} u_i \cdot z_i - \sum_{ij \in \left(\mathcal{I}'_0\right)^2} \mu_{ij} \cdot c_{ij}$$

Note that now, as $\forall ij \in \mathcal{I}^2$, $p_{j,ij} - p_{i,ij} - R_{ij}\left(p_{i,ij}, p_{j,ij}\right) = c_{ij}$ we know that if $R_{ij}\left(p_{i,ij}, p_{j,ij}\right) = 0$ then $\left(\nabla p\right)_{ij} = c_{ij}$. Then V_μ is constant and does not depend on the price anymore.

Each agent has a participation constraint $\forall i \in \mathcal{I}$, $\left(\nabla^* \mu p\right)_i \geq 0$ and $\forall i \in \mathcal{I}' \cup \{0\}$, $\left(\nabla^* \mu p\right)_i = 0$ so efficient allocations of the surplus verify:

$$\begin{cases} \sum_{i \in \mathcal{I}'_0} \left(\nabla^* \mu p\right)_i = V_\mu \\ \forall i \in \mathcal{I},\ \left(\nabla^* \mu p\right)_i \geq 0 \\ \forall i \in \mathcal{I}' \cup \{0\},\ \left(\nabla^* \mu p\right)_i = 0 \end{cases}$$

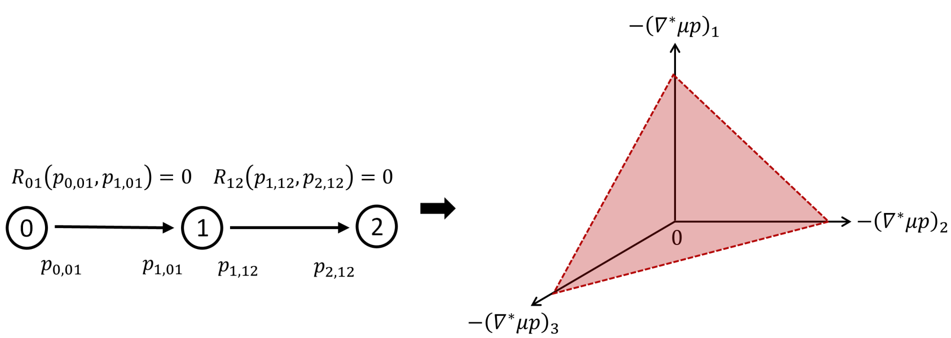

FIGURE 5.1. Example of bargaining set

The Shapley value is the only distribution of surplus that verifies the four assumptions: efficiency, symmetry, linearity, null surplus for a zero player. It is a "fair" allocation of the surplus generated in the sens that it reflects the true participation of a player to the coalition. $\forall i \in \mathcal{I}'_0$, the Shapley value is:

$$\phi_i = \sum_{S \subseteq \mathcal{I} - \{i\}} \frac{|S|!\left(N_\mathcal{I} - |S| - 1\right)!}{N_\mathcal{I}!} \left(v\left(S \cup \{i\}\right) - v\left(S\right)\right)$$

For $\mu \in \mathscr{W}_0$, the associated price is p_ϕ such that:

$$\begin{cases} \forall i \in \mathcal{I},\ \left(\nabla^* \mu p_\phi\right)_i = \phi_i \\ \forall i \in \mathcal{I}' \cup \{0\},\ \left(\nabla^* \mu p\right)_i = 0 \\ \forall ij \in \left(\mathcal{I}'_0\right) \text{ s.t. } \mu_{ij} > 0,\ \left(\nabla p_\phi\right)_{ij} = c_{ij} \end{cases}$$

Proposition. *v is supermodular so the game is convex and the Shapley value is the center of gravity of the core.*

6. An application: the overnight interbank loan market

6.1. **Presentation of the market.** The overnight interbank loan market is a decentralized, highly liquid market that funds temporary and localized needs of liquidity. Because of its role, it contributes to the efficiency of the banking system. It is also an instrument of monetary policy through the setting by central banks of the interest rates corridor. Furthermore the rate of interbank loans is an important guide for other loans and for the pricing of bonds and equities.

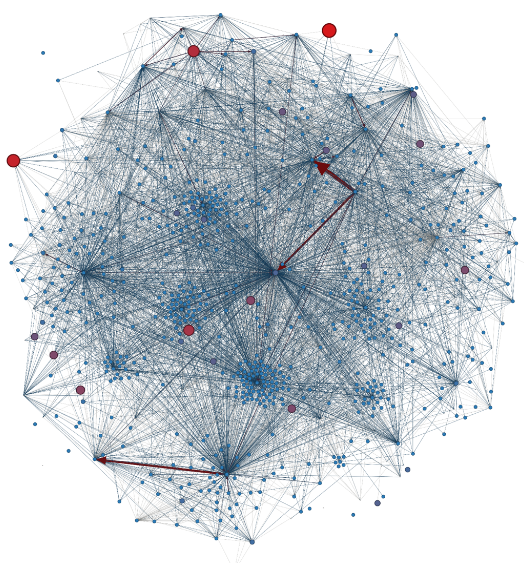

FIGURE 6.1. Overnight interbank loan market (Austria, 03/2011) - Data: OeNB

During crises, for example in 2007-2008, the liquidity of the overnight interbank loan market can quickly dry, leaving the role of clearing house to the central bank. Loans contracted by banks can spread and amplify shocks on the financial system, making the interbank loan market one of the channel of systemic risk.

In our model we consider the overnight interbank loan market as a decentralized market on which the only traded commodity is liquidity (i.e. money market instruments: certificate of deposit, repurchase agreements, commercial paper, ...). The numeraire is liquidity in the following morning and prices are the overnight interest rates.

Each bank aims at reaching, by the end of a business day the level of liquidity required by the regulator. Random shocks affect its level of liquidity and make it enter the interbank loan market as a borrower or a lender. Then a bank faces a discrete choice between several loan partners offering different rates and different risk of default. All banks have an outside option that is offered by the central bank: it can borrow at the repo rate or lend at the deposit interest rate. Those arguments tend to justify the use of the utility functions presented in section 4.

6.2. **Setting.** $\mathcal{I}$ is the set of banks participating in the overnight interbank loan market. We have one commodity (liquidity in t) and one numeraire (repayment in $t+1$). For a loan of size μ_{ij} between agent i and agent j, a part $\beta_{ij} \cdot \mu_{ij}$ is non-collateralized.

Utility is imperfectly transferable because of:

(1) Counterparty risk. We introduce $d_{ij} \in \{0;1\}$ a random variable, revealed after the loan, that describes if j defaults on non-collateralized part of the loan granted by i.
(2) Risk aversion. Traders working for bank i lend to $\underset{j \text{ st } (ij)\in\mathcal{I}^2}{\arg\max}\, \mathbb{E}\left[W_i\left(\left(1-d_{ij}\right)\cdot p_j\right)\right]$ with W_i an increasing and concave function.
(3) Collateralization costs. Borrowers have to collateralize part of their loans. This fact of immobilizing those asset has a marginal cost c_{ij} (risk of losses of value for the collateral for example).

Hence $\forall ij \in \mathcal{I}^2$,

$$R_{ij}\left(p_i,p_j\right) = \left(1-\beta_{ij}\right)\cdot p_j + \beta_{ij}\cdot \mathbb{E}\left[W_i\left(\left(1-d_{ij}\right)\cdot p_j\right)\right] - p_i - \left(1-\beta_{ij}\right)\cdot c_{ij}$$

An application of the model would be the quantification of the risk aversion in the interbank loan market. From the observation of flows and prices, we would estimate the parameters of functions R_{ij}.

7. Conclusion

We showed in this paper that, if we allow utility to be imperfectly transferable between agents, competitive equilibrium can be reached through bipartite contracting in large multi-commodity markets. One-to one and one-to-many matching problems are particular cases of our model. We also gave computation methods to find a partial equilibrium outcome.

We are working on several developments of our results. A natural extension would incorporate production and we intend to investigate the links between equilibrium flows and gravity equations. We also plan to adapt Scarf's algorithm to allow for the computation of equilibrium outcomes in the general case.

Finally, if we discuss the efficiency of allocations in this article, we did not raise the question of stability. Future work on the subject should focus on characterizing equilibrium and stability seems the natural next challenge.

Appendix. Algorithms

Generalized simplex algorithm.

Definition. A few definitions of terms used in the algorithm:

(1) A tree is a connected acyclic sub-graph.
(2) A spanning tree of G is a tree that includes all the nodes of G.
(3) A leaf node of a tree is a node with one incident arc.
(4) A spanning tree is said feasible if there exists a feasible flow on this tree.
(5) A spanning tree is strongly feasible if all the arcs of the tree with a flow equal to zero is oriented away from the root 0.
(6) The simplex algorithm generates a succession of strongly feasible trees $\left(T^k\right)_{k\in\mathbb{N}}$. At iteration k, the in-arc is the arc that belongs to T^k but not to T^{k-1} and the out-arc is the arc that belongs to T^{k+1} but not to T^k.

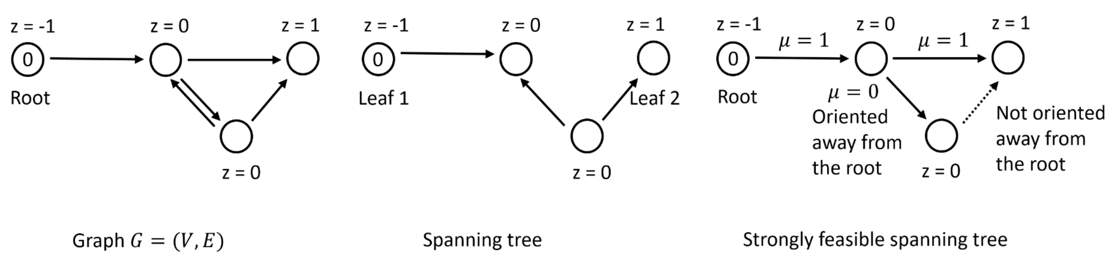

Figure 7.1. Trees

Big-M transformation.
We choose M large and add an agent m such that $\forall i \in \mathcal{I}_0'$, $R_{mi}(p_m, p_i) = p_i - p_m - M$ and $R_{im}(p_i, p_m) = p_m - p_i - M$.

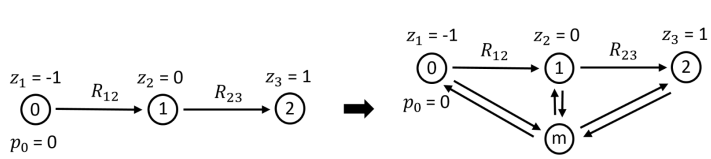

Figure 7.2. Big-M transformation

Initialization.

(1) We start from a graph $G_0 = \left(\mathcal{I}_0', (\mathcal{I}_0')^2\right)$ and apply the big-M transformation described above.
(2) We initialize the algorithm by choosing a strongly feasible spanning tree T^0 on this graph such that if $z_i \geq 0$ $\mu^0_{mi} = z_i$, if $z_{i'} < 0$ $\mu^0_{im} = -z_i$ and a flow null everywhere else. We define on T^0, the associated feasible flow μ^0 and a price vector p^0 such that

$$\begin{cases} p^0_{i=0} = 0 \\ \forall ij \in T^0,\ R_{ij}\left(p^0_i, p^0_j\right) = 0 \end{cases}$$

p^0 and μ^0 are unique.

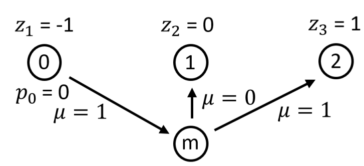

FIGURE 7.3. Generalized simplex algorithm - Initialization

Recurrence.

(1) At iteration $k > 0$, we have a strongly feasible spanning tree T^k, μ^k is the associated feasible flow and p^k the price vector such that

$$\begin{cases} p^k_{i=0} = 0 \\ \forall ij \in T^k,\ R_{ij}\left(p^k_i, p^k_j\right) = 0 \end{cases}$$

(2) We select an in-arc $\overline{e}^k = ij \in (\mathcal{I}_0')^2$ such that $R_{ij}\left(p^k_i, p^k_j\right) > 0$. Immediately $\overline{e}^k \notin T^k$. If we can't find one the algorithm terminates, $\left(\mu^k, p^k\right)$ is an equilibrium outcome.

Adding $\overline{e}^k$ to T^k we add a cycle C^k. C^{k+} is the set of edges of C^k oriented in the same direction as $\overline{e}^k$ and C^{k-} those oriented in the opposite direction.

We select the out-arc $\underline{e}^k = \underset{e \in C^{k-}}{\text{argmin}} \mu_e$. If C^{k-} is empty there exists a cycle of negative cost.

(3) We change the flow according to

$$\mu^{k+1} = \begin{cases} \forall e \in C^{k-},\ \mu_e^k - \min\limits_{e \in C^{k-}} \mu_e^k \\ \forall e \in C^{k+},\ \mu_e^k + \min\limits_{e \in C^{k-}} \mu_e^k \\ \forall e \notin C^{k-} \cup C^{k+},\ \mu_e^k \end{cases}$$

The tree $T^{k+1} = T^k - \underline{e}^k + \overline{e}^k$ is a strongly feasible spanning tree associated to μ^{k+1}.

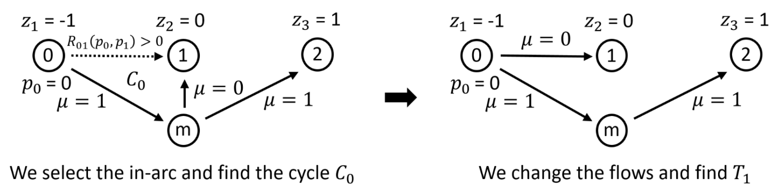

FIGURE 7.4. Generalized simplex algorithm - Recurrence

Termination.
We remove node m and all arcs of which it is a member

Decentralized auction algorithm.

Definition. A few definitions of terms used in the algorithm:

(1) The set of ϵ-equilibrium outcomes is:

$$\mathscr{E}_\epsilon = \left\{ (\mu, p) \text{ s.t. } \begin{array}{l} \mu \in \mathscr{W}_0 \\ \forall ij \in (\mathcal{I}_0')^2,\ R_{ij}(p_i, p_j) \leq \epsilon \\ \forall ij \in (\mathcal{I}_0')^2 \text{ s.t. } \mu_{ij} > 0,\ -\epsilon \leq R_{ij}(p_i, p_j) \leq \epsilon \end{array} \right\}$$

(2) An arc $ij \in (\mathcal{I}_0')^2$ is ϵ^+-unblocked if $R_{ij}(p_i, p_j) = \epsilon$.
(3) An arc $ij \in (\mathcal{I}_0')^2$ is said ϵ^--unblocked if $R_{ij}(p_i, p_j) = -\epsilon$ and $\mu_{ij} > 0$.
(4) The candidate list of a node i is the set of outgoing edges ϵ^+-unblocked and the incoming edges ϵ^--unblocked.
(5) The excess demand after flows in i is $g_i = z_i - (\nabla^* \mu)_i$.

Initialization.
(μ^0, p^0) that satisfy the ϵ-complementary slackness conditions:

$$\begin{cases} \forall ij \in (\mathcal{I}_0')^2, \ R_{ij}\left(p_i^0, p_j^0\right) \leq \epsilon \\ \forall ij \in (\mathcal{I}_0')^2 \text{ s.t. } \mu_{ij}^0 > 0, \ -\epsilon \leq R_{ij}\left(p_i^0, p_j^0\right) \leq \epsilon \end{cases}$$

Recurrence.
At iteration $k > 0$, we have $\left(\mu^k, p^k\right)$ that verifies:

$$\begin{cases} \forall ij \in (\mathcal{I}_0')^2, \ \mu_{ij}^k \geq 0 \\ \forall ij \in (\mathcal{I}_0')^2, \ R_{ij}\left(p_i^k, p_j^k\right) \leq \epsilon \\ \forall ij \in (\mathcal{I}_0')^2 \text{ s.t. } \mu_{ij}^k > 0, \ -\epsilon \leq R_{ij}\left(p_i^k, p_j^k\right) \leq \epsilon \end{cases}$$

Part 1. We select $i \neq 0$ such that $g_i^k < 0$. If there does not exist such i we move to the next part of the mechanism.

(1) If the candidate list of i is empty go to 4. If not select an outgoing arc ij and go to 2 or select an arc ji and go to 3.
(2) Increase μ_{ij}^k by $\delta = -g_i^k$. Next iteration.
(3) Decrease μ_{ji}^k by $\delta = \min\left(-g_i^k, \mu_{ji}^k\right)$. If $g_i^k = 0$ next iteration, else go back to 1.
(4) Decrease p_i^k to $\underline{p_i^k}$ and go back to 1.

$$\underline{p_i^k} = \max \quad \Big\{\max_r \left\{r \text{ s.t. } \exists\, (ji) \in (\mathcal{I}_0')^2 \text{ and } \mu_{ji} > 0, \ R_{ji}\left(p_j^k, r\right) = -\epsilon\right\}, \\ \max_r \left\{r \text{ s.t. } \exists\, (ij) \in (\mathcal{I}_0')^2, \ R_{ij}\left(r, p_j^k\right) = \epsilon\right\}\Big\}$$

Part 2. We select $i \neq 0$ such that $g_i^k > 0$. If there does not exist such i the algorithm terminates and $\left(\mu^k, p^k\right) \in \mathscr{E}_\epsilon$.

(1) If the candidate list of i is empty go to 4. If not select an outgoing arc ij and go to 2 or select an arc ji and go to 3.
(2) Decrease μ_{ij}^k by $\delta = \min\left(g_i^k, \mu_{ij}^k\right)$. If $g_i = 0$ next iteration, else go back to 1.
(3) Increase μ_{ji}^k by $\delta = g_i^k$. Next iteration.
(4) Increase p_i^k to $\overline{p_i^k}$ and go back to 1.

$$\overline{p_i^k} = \min \quad \Big\{\min_r \left\{r \text{ s.t. } \exists\, (ij) \in (\mathcal{I}_0')^2 \text{ and } \mu_{ij} > 0, \ R_{ij}\left(r, p_j\right) = \epsilon\right\}, \\ \min_r \left\{r \text{ s.t. } \exists\, (ji) \in (\mathcal{I}_0')^2, \ R_{ji}\left(p_j, r\right) = -\epsilon\right\}\Big\}$$

Appendix. Proofs

Existence of an equilibrium outcome.

Proof. By iteration $\forall i, l$ we build $\hat{P} \in \mathbb{R}_+^{N_{\mathcal{I}} \times N_{\mathcal{L}}}$ such that:

(1) $\forall (i,l),\ \hat{P}_{i,l}^1 = u_{i,l}$.
(2) $\hat{P}_{i,l}^2 = \max\left\{\hat{P}_{i,l}^1; \forall j \in \mathcal{I},\ r \text{ s.t. } R_{ij,l}\left(r, \hat{P}_{j,l}^1\right) = 0\right\}$.
(3) ...

(k) $\hat{P}_{i,l}^k = \max\left\{\hat{P}_{i,l}^{k-1}; \forall j \in \mathcal{I},\ r \text{ s.t. } R_{ij,l}\left(r, \hat{P}_{j,l}^k\right) = 0\right\}$.

We stop iterating after $N_{\mathcal{I}+1}$ iterations, $\hat{P} = \hat{P}^{N_{\mathcal{I}}+1}$

We introduce the non-empty, compact and convex sets:

$$
\begin{aligned}
M &= \left\{\mu \in \mathbb{R}_+^{N_{\mathcal{I}}^2 \times N_{\mathcal{L}}} \text{ s.t } \forall (ij,l) \in \mathcal{I}^2 \times \mathcal{L},\ \mu_{ij,l} \leq 2 \cdot \sum_i e_{i,l}\right\} \\
P &= \left\{p \in \mathbb{R}_+^{N_{\mathcal{I}} \times N_{\mathcal{L}}} \text{ s.t } \forall (i,l) \in \mathcal{I} \times \mathcal{L},\ p_{i,l} \leq \hat{P}_{i,l}\right\} \\
Q &= \left\{q \in \mathbb{R}_+^{N_{\mathcal{I}} \times N_{\mathcal{L}}} \text{ s.t } \forall (i,l) \in \mathcal{I} \times \mathcal{L},\ q_{i,l} \leq 2 \cdot \sum_i e_{i,l}\right\}
\end{aligned}
$$

We define the non-empty, convex-valued correspondences:

(1) The demand correspondence:

$$\Psi^D(p) = \left[\underset{q_i \in Q}{\arg\max}\left(V_i\left(q_i + e_i\right) - q_i \cdot p_i\right)\right]_{i \in \mathcal{I}}$$

(2) The trade correspondence:

$$\Psi^T(p) = \underset{\mu \in M}{\arg\max}\left(\sum_{ij,l} \mu_{ij,l} \cdot R_{ij,l}\left(p_{i,l}, p_{j,l}\right)\right)$$

(3) The price correspondence:

$$\Psi^P(\mu, q) = \underset{p \in P}{\arg\max}\left(\sum_{i,l}\left(q_{i,l} - \left(\nabla^* \mu_l\right)_i\right) \cdot p_{i,l}\right)$$

The maximum theorem gives us that and those correspondence are upper-semi-continuous.

We now consider the non-empty, convex-valued and upper-semi-continuous correspondence:

$$
\Psi : \begin{array}{ccc} M \times P \times Q & \rightarrow & M \times P \times Q \\ (\mu, p, q) & \mapsto & \left(\Psi^T(p), \Psi^P(\mu, q), \Psi^D(p)\right) \end{array}
$$

Kakutani fixed point theorem gives $\exists\left(\mu^{*}, p^{*}, q^{*}\right) \in K \times S \times U$ s.t. $\left(\mu^{*}, p^{*}, q^{*}\right) \in \Psi\left(\mu^{*}, p^{*}, q^{*}\right)$.

As $p^{*} \in \Psi^{P}\left(\mu^{*}, q^{*}\right)$ so:

- If $q_{i,l}^{*}=\left(\nabla^{*} \mu_{l}^{*}\right)_{i}$ then we immediately have $\nabla^{*} \mu^{*} \in \Psi^{D}\left(p^{*}\right)$ and $\mu^{*} \in \mathscr{W}\left(p^{*}\right)$.
- If $q_{i,l}^{*}<\left(\nabla^{*} \mu_{l}^{*}\right)_{i}$ then $p_{i,l}^{*}=0$. If there is no-satiation of agent i in commodity l, this is not a fixed point as $q^{*} \notin \Psi^{D}\left(p^{*}\right)$ and we have a contradiction. If we have satiation at some point then as $q_{i,l}^{*} \in \Psi^{D}\left(p^{*}\right)$ we also have $\nabla^{*} \mu^{*} \in \Psi^{D}\left(p^{*}\right)$ and $\mu^{*} \in \mathscr{W}\left(p^{*}\right)$.
- If $q_{i,l}^{*}>\left(\nabla^{*} \mu_{l}^{*}\right)_{i}$ then $p_{i,l}^{*}=\hat{P}_{i,l}$. As $q^{*} \in \Psi^{D}\left(p^{*}\right)$, by definition of $\hat{P}$, $p_{i,l}^{*} \geq u_{i,l}$, $q_{i,l}^{*}=-e_{i,l} \leq 0$ and so $\left(\nabla^{*} \mu_{l}^{*}\right)_{i}<0$.
  - If $\forall j,\ R_{ij,l}\left(p_{i,l}^{*}, p_{j,l}^{*}\right)<0$ then as $\mu^{*} \in \Psi^{T}\left(p^{*}\right)$, $\left(\nabla^{*} \mu_{l}^{*}\right)_{i} \geq-e_{i,l} \geq 0$. Contradiction.
  - Else $\forall j$ s.t. $R_{ij,l}\left(p_{i,l}^{*}, p_{j,l}^{*}\right) \geq 0$ by definition of $\hat{P}$ we have $p_{j,l}^{*} \geq u_{j,l}$. As $q^{*} \in \Psi^{D}\left(p^{*}\right)$, $q_{j,l}^{*}=-e_{j,l}$. As $\left(\nabla^{*} \mu_{l}^{*}\right)_{i}<0$, $\exists j$ such that $\mu_{ij,l}>0$. Either j gives the last contradiction or there exists a z that verifies the same property. If we don't reach a agent with a contradiction, then we contradict the assumption of no existence of a cycle of negative cost.

Hence $\mu^{*} \in \mathscr{W}\left(p^{*}\right)$.

As $\mu^{*} \in \Psi^{T}\left(p^{*}\right)$, if $R_{ij,l}\left(p_{i}^{*}, p_{j}^{*}\right)<0$ then $\mu_{ij,l}^{*}=0$ and if $R_{ij,l}\left(p_{i}^{*}, p_{j}^{*}\right)>0$ then $\mu_{ij,l}^{*}=2 \cdot \sum_{i} e_{i,l}$. We showed by contradiction a few line above that $q_{i,l}^{*} \leq \nabla^{*} \mu^{*}$, as there does not exist a cycle of negative cost $\mu_{ij,l}^{*} \leq \sum_{i} e_{i,l}<2 \cdot \sum_{i} e_{i,l}$ so $R_{ij,l}\left(p_{i}^{*}, p_{j}^{*}\right) \leq 0$. Hence $p^{*} \in \mathscr{S}\left(\mu^{*}\right)$. □

Remark. If $R_{ij,l}\left(p_{i,l}, p_{j,l}, \mu_{ij,l}\right)$ depends on $\mu_{ij,l}$ (congestion, ...), this proof can be easily extended as long as $\mu_{ij,l} \cdot R_{ij,l}\left(p_{i,l}, p_{j,l}, \mu_{ij,l}\right)$ is concave. Hence if

$$\frac{\partial^{2} \mu_{ij,l} \cdot R_{ij,l}}{\partial \mu_{ij,l}^{2}}=2 \frac{\partial R_{ij,l}}{\partial \mu_{ij,l}}+\mu_{ij,l} \cdot \frac{\partial^{2} R_{ij,l}}{\partial \mu_{ij,l}^{2}} \leq 0$$

Welfare theorems.

Proof. **Counter example.** $(\mu, p) \in \mathscr{E}$ may not be Pareto efficient.

Consider the following economy: two agents $\mathcal{I} = \{i, j\}$, two commodities $\mathcal{L} = \{l_1, l_2\}$ and one numeraire. The vectors of initial endowments are

$$e_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } e_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

utility functions for the consumption of commodities are

$$\begin{cases} V_i\left(q_{l_1}, q_{l_2}\right) = 100 \cdot \max\left(q_{l_2}; 1\right) \\ V_i\left(q_{l_1}, q_{l_2}\right) = 100 \cdot \max\left(q_{l_1}; 1\right) \end{cases}$$

and there is an import duty on the transfer of commodity $t_{ij} = t_{ji} = 0.1$, so the transfer functions are

$$\begin{cases} R_{ij,l_1}\left(p_{i,l_1}, p_{j,l_1}\right) = p_{j,l_1} - p_{i,l_1} - 0.1 \cdot p_{i,l_1} \\ R_{ji,l_2}\left(p_{j,l_2}, p_{i,l_2}\right) = p_{i,l_2} - p_{j,l_2} - 0.1 \cdot p_{j,l_2} \end{cases}$$

Hence the following outcome is an equilibrium

$$p_i = \begin{bmatrix} 10 \\ 11 \end{bmatrix},\ p_j = \begin{bmatrix} 11 \\ 10 \end{bmatrix},\ \mu_{ij,l_1} = 1 \text{ and } \mu_{ji,l_2} = 1$$

but this outcome is strictly preferred by all agents

$$p_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ p_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mu_{ij,l_1} = 1 \text{ and } \mu_{ji,l_2} = 1$$

First welfare theorem. However if $\forall\,(ij, l) \in \mathcal{I}^2 \times \mathcal{L}$ the quantity $\left[p_{j,l} - p_{i,l} - R_{ij,l}\left(p_{i,l}, p_{j,l}\right)\right]$ does not depend on prices then for $(\mu, p) \in \mathscr{E}$ it does not exist (μ', p') such that $\forall i \in \mathcal{I}$, $V_i\left(\left(\nabla^*\mu'\right)_i + e_i\right) - \left(\nabla^*\mu'\right)_i \cdot p'_i \geq V_i\left(\left(\nabla^*\mu\right)_i + e_i\right) - \left(\nabla^*\mu\right)_i \cdot p_i$ and $\exists i$ such that $V_i\left(\left(\nabla^*\mu'\right)_i + e_i\right) - \left(\nabla^*\mu'\right)_i \cdot p'_i > V_i\left(\left(\nabla^*\mu\right)_i + e_i\right) - \left(\nabla^*\mu\right)_i \cdot p_i$.

We said that p' is feasible if $\forall\,(ij, l)$ such that $\mu'_{ij,l} > 0$ we have $R_{ij,l}\left(p'_{i,l}, p'_{j,l}\right) \geq 0$.

By contradiction let's assume that $(\mu, p) \in \mathscr{E}$ and that $\exists\,(\mu', p')$ such that $\forall i \in \mathcal{I}$, $V_i\left(\left(\nabla^*\mu'\right)_i + e_i\right) - \left(\nabla^*\mu'\right)_i \cdot p'_i \geq V_i\left(\left(\nabla^*\mu\right)_i + e_i\right) - \left(\nabla^*\mu\right)_i \cdot p_i$ and $\exists i$ such that $V_i\left(\left(\nabla^*\mu'\right)_i + e_i\right) - \left(\nabla^*\mu'\right)_i \cdot p'_i > V_i\left(\left(\nabla^*\mu\right)_i + e_i\right) - \left(\nabla^*\mu\right)_i \cdot p_i$. $\forall\,(ij, l) \in \mathcal{I}^2 \times \mathcal{L}$ we define $c_{ij,l} = p_{j,l} - p_{i,l} - R_{ij,l}\left(p_{i,l}, p_{j,l}\right)$.

So, as $\mu \in \mathscr{W}(p)$, by revealed preference, we must have $\sum\limits_{i,l}\left[\left(\nabla^*\mu'_l\right)_i - \left(\nabla^*\mu_l\right)_i\right] \cdot p_{i,l} - \sum\limits_{i,l}\left[\left(\nabla^*\mu'_l\right)_i \cdot p'_{i,l} - \left(\nabla^*\mu_l\right)_i \cdot p_{i,l}\right] > 0$. Then $\sum\limits_{i,l}\left(\nabla^*\mu'_l\right)_i \cdot p_{i,l} > \sum\limits_{i,l}\left(\nabla^*\mu'_l\right)_i \cdot p'_{i,l}$ which can be rewritten as $\sum_{ij,l}\mu'_{ij,l}\cdot(p_{j,l}-p_{i,l})>\sum_{ij,l}\mu'{}_{ij,l}\cdot\left(p'_{j,l}-p'{}_{i,l}\right)$ and $\sum_{ij,l}\mu'_{ij,l}\cdot(c_{ij,l}+R_{ij,l}(p_{i,l},p_{j,l}))>\sum_{ij,l}\mu'{}_{ij,l}\cdot\left(c_{ij,l}+R_{ij,l}\left(p'_{i,l},p'_{j,l}\right)\right)$. So we should have $\sum_{ij,l}\mu'_{ij,l}\cdot R_{ij,l}(p_{i,l},p_{j,l})>\sum_{ij,l}\mu'{}_{ij,l}\cdot R_{ij,l}\left(p'_{i,l},p'_{j,l}\right)$. However, as $p\in\mathscr{S}(\mu)$, $\sum_{ij,l}\mu'{}_{ij,l}\cdot R_{ij,l}(p_{i,l},p_{j,l})\leq 0$ so $\sum_{ij,l}\mu'{}_{ij,l}\cdot R_{ij,l}\left(p'_{i,l},p'_{j,l}\right)<0$, p' is not feasible and we have a contradiction.

Second welfare theorem. If the initial endowment e is Pareto efficient and there is no trade subsidies, $\forall(ij,l)\in\mathcal{I}^2\times\mathcal{L}$, $p_{j,l}-p_{i,l}-R_{ij,l}(p_{i,l},p_{j,l})\geq 0$, then there exists a price vector p such that $\left(\widetilde{0},p\right)\in\mathscr{E}$ ($\widetilde{0}$ is the flow null).

The second welfare theorem formulation of Arrow and Debreu (1951) holds without any modification. If e is Pareto efficient then there exists $(p_l)_{l\in C}$ s.t. if $\forall(i,l)\in\mathcal{I}\times\mathcal{L}$, $p_{i,l}=p_l$ and $\widetilde{0}\in\mathscr{W}(p)$. By hypothesis $\forall(ij,l,l')\in\mathcal{I}^2\times\mathcal{L}\times\mathcal{L}$, $p_{j,l}-p_{i,l}-R_{ij,l}(p_{i,l},p_{j,l})=-R_{ij,l}(p_{i,l},p_{j,l})\geq 0$. So $p\in\mathscr{S}\left(\widetilde{0}\right)$ and $\left(\widetilde{0},p\right)\in\mathscr{E}$. □

Min-cost flow duality.

Proof. Duality of the min-cost flow problem:

$$
\begin{aligned}
&\min_{\mu_{ij}\geq 0\text{ s.t. }\mu\in\mathscr{W}_0}\left(\sum_{ij}\mu_{ij}\cdot c_{ij}\right)\\
&=\min_{\mu_{ij}\geq 0\text{ s.t. }\forall i\in\mathcal{I}'_0,\ (\nabla^*\mu)_i=z_i}\left(\sum_{ij}\mu_{ij}\cdot c_{ij}\right)\\
&=\min_{\mu_{ij}\geq 0,\ p_i}\left(\sum_{ij}\mu_{ij}\cdot c_{ij}+\sum_i p_i\cdot\left(z_i-(\nabla^*\mu)_i\right)\right)\\
&=\min_{\mu_{ij}\geq 0,\ p_i}\left(\sum_i p_i\cdot z_i+\sum_{ij}\mu_{ij}\cdot c_{ij}-\sum_i p_i\cdot(\nabla^*\mu)_i\right)\\
&=\min_{\mu_{ij}\geq 0,\ p_i}\left(\sum_i p_i\cdot z_i+\sum_{ij}\mu_{ij}\cdot\left(c_{ij}-(\nabla p)_{ij}\right)\right)\\
&=\min_{p\text{ s.t. }\forall ij\in\left(\mathcal{I}'_0\right)^2,\ (\nabla p)_{ij}\leq c_{ij}}\left(\sum_i p_i\cdot z_i\right)
\end{aligned}
$$

First order conditions give the complementary slackness conditions:

$$
\begin{cases}
\mu\in\mathscr{W}_0\\
\forall ij\in\left(\mathcal{I}'_0\right)^2,\ (\nabla p)_{ij}-c_{ij}\leq 0\\
\forall ij\in\left(\mathcal{I}'_0\right)^2\text{ s.t. }\mu_{ij}>0,\ (\nabla p)_{ij}-c_{ij}=0
\end{cases}
$$

So $\mu\in\mathscr{C}\Leftrightarrow\exists p$ s.t. $(\mu,p)\in\mathscr{E}$. □

Generalized simplex algorithm.

Proof. **Selection of the value for M.** We need to set M to a large enough value so that the flow on those edges is null at equilibrium. Let T be a spanning tree of G, $\forall i \in V$ we define p_i^T the price in i such that $\begin{cases} p_{i^*} = p^* \\ \forall (ij) \in T,\ R_{ij}(p_i, p_j) = 0 \end{cases}$. There exists a finite number of spanning tree of G, we note $\begin{cases} p_{max} = \max\limits_{i,\ T} p_i^T \\ p_{min} = \min\limits_{i,\ T} p_i^T \end{cases}$. Finally we take $M > \frac{(N_v-1)(p_{max}-p_{min})}{2}$.

The algorithm terminates. $G = \left(\mathcal{I}_0', (\mathcal{I}_0')^2\right)$. $\forall S \subseteq \mathcal{I}_0'$ we define

$$\overline{S} = \left\{ i \in \mathcal{I}_0' \text{ s.t. } \begin{array}{l} \text{either } i \in S \\ \text{or } \exists j \in \overline{S} \text{ s.t. } ji \in (\mathcal{I}_0')^2 \text{ and isn't involved in any cycle} \end{array} \right\}$$

$G_1 = (S_1, S_1^2)$, ... , $G_m = (S_m, S_m^2)$ sub-graphs of G such that

$$\begin{cases} G_1 = \bigcup\limits_{C \in \{\text{cycle containing } 0\}} C \\ \dots \\ G_m = \bigcup\limits_{\substack{C \in \left\{\text{cycle containing an } i \in \overline{S_{m-1}}\right\} \\ C \notin G_{m-1}}} C \end{cases}$$

At each iteration we have

$$\begin{cases} \text{either } \sum\limits_{e \in G_1} R_e(p_k) < \sum\limits_{e \in G_1} R_e(p_{k-1}) \\ \text{or } \sum\limits_{e \in G_1} R_e(p_k) = \sum\limits_{e \in G_1} R_e(p_{k-1}) \text{ and} \begin{cases} \text{either } \sum\limits_{e \in G_2} R_e(p_k) < \sum\limits_{e \in G_2} R_e(p_{k-1}) \\ \text{or } \sum\limits_{e \in G_2} R_e(p_k) = \sum\limits_{e \in G_2} R_e(p_{k-1}) \text{ and...} \end{cases} \end{cases}$$

All trees generated by the algorithm are different and there exists a finite number of strongly feasible spanning tree.

The algorithm terminates after a finite number of iteration. Furthermore we get an integer flow.

If the equilibrium flow is non-null on edges added during the big-M transformation then $\mathscr{F} = \emptyset$ and if the problem is unbounded on the big-M version of G then G is unbounded. Let's assume we have an equilibrium flow with non-null component on the edges added during the big-M transformation. Let's take for each edge $e \in E$ the constant marginal cost $c_e = (\nabla p)_e$ and we get our result for the network with linear cost from Bertsekas (1998), proposition 5.5. ☐

Decentralized auction algorithm.

Proof. **Step 1.** Show that if there does not exist a cycle of negative cost the part 1 of the algorithm terminates.

By contradiction : let's assume part 1 of the recurrence doesn't terminate.

$\exists ij \in (\mathcal{I}'_0)^2$ with an infinite number of changes of flow, alternatively ϵ^+-blocked with $g_i < 0$ and ϵ^--blocked with $g_j < 0$.

$(\nabla p)_{ij} - c_{ij}(p) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ so $p_i \longrightarrow -\infty$ and $p_j \longrightarrow -\infty$. Contradiction.

Step 2. Show that if there does not exist a cycle of negative cost the part 2 of the algorithm terminates.

By contradiction : let's assume part 2 of the recurrence doesn't terminate.

$\exists ij \in (\mathcal{I}'_0)^2$ with an infinite number of changes of flow, alternatively ϵ^--blocked with $g_i > 0$ and ϵ^+-blocked with $g_j > 0$.

$(\nabla p)_{ij} - c_{ij}(p) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ so $p_i \longrightarrow +\infty$ and $p_j \longrightarrow +\infty$. $\mathcal{I}_0'^{+\infty} \neq \emptyset$.

By ϵ-complementary slackness conditions : $\exists! (i,j) \in A_0'^{+\infty} \times \left(A'_0 - A_0'^{+\infty}\right)$ such that $ij \in (A'_0)^2$

If $(i,j) \in \left(\mathcal{I}'_0 - \mathcal{I}_0'^{+\infty}\right) \times \mathcal{I}_0'^{+\infty}$ such that $ij \in (\mathcal{I}'_0)^2$ then $\mu_{ij} = 0$.

$\sum_{i \in \mathcal{I}_0'^{-\infty}} g_i < 0$ with no outgoing link from $\mathcal{I}_0'^{+\infty}$. So $\mathscr{F} = \emptyset$ and we know that there exists one.

Contradiction, so the algorithm terminates.

Step 3. We now show that if $\epsilon > 0$ is small enough and μ satisfies the ϵ-complementary slackness conditions then the algorithm converges toward μ that is an equilibrium flow.

We choose (for T strongly feasible trees)

$$\epsilon < \min_T \left\{ \min_{\substack{ij \in (\mathcal{I}'_0)^2 \text{ st } (\nabla p)_{ij} - c_{ij}(p_i, p_j) > 0 \\ p \text{ st } \forall e \in T,\ (\nabla p)_e - c_e(p) = 0}} \left\{ (\nabla p)_{ij} - c_{ij}(p_i, p_j) \right\} \right\}$$

By contradiction : we assume $(\mu, p) \in \mathscr{E}_\epsilon$ and μ isn't an equilibrium flow. Then we know that: $\exists ij \in (\mathcal{I}'_0)^2$ such that $(\nabla p)_{ij} - c_{ij}(p_i, p_j) > 0$.

And $\forall ij \in (\mathcal{I}'_0)^2$ such that $(\nabla p)_{ij} - c_{ij}(p_i, p_j) > 0$, $(\nabla p)_{ij} - c_{ij}(p_i, p_j) \leq \epsilon$.

Because of the value of ϵ no tree can bear this flow. Contradiction. □

Convexity of the game

Proof. $\forall S_1, S_2 \subseteq \mathcal{I}$:

$$\begin{aligned}
& v(S_1 \cap S_2) + v(S_1 \cup S_2) \\
&= \sum_{i \in (S_1 \cap S_2) \cup \{0\} \text{s.t.} z_i > 0} u_i \cdot z_i - \min_{\mu_{|(S_1 \cap S_2) \cup \{0\}} \in \mathscr{W}_{(S_1 \cap S_2) \cup \{0\}}} \left[\sum_{ij \in ((S_1 \cap S_2) \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&\quad + \sum_{i \in S_1 \cup S_2 \cup \{0\} \text{ s.t. } z_i > 0} u_i \cdot z_i - \min_{\mu_{|S_1 \cup S_2 \cup \{0\}} \in \mathscr{W}_{S_1 \cup S_2 \cup \{0\}}} \left[\sum_{ij \in (S_1 \cup S_2 \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&= \sum_{i \in S_1 \cup \{0\} \text{ s.t. } z_i > 0} u_i \cdot z_i - \min_{\mu_{|(S_1 \cap S_2) \cup \{0\}} \in \mathscr{W}_{(S_1 \cap S_2) \cup \{0\}}} \left[\sum_{ij \in ((S_1 \cap S_2) \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&\quad + \sum_{i \in S_2 \cup \{0\} \text{ s.t. } z_i > 0} u_i \cdot z_i - \min_{\mu_{|S_1 \cup S_2 \cup \{0\}} \in \mathscr{W}_{S_1 \cup S_2 \cup \{0\}}} \left[\sum_{ij \in (S_1 \cup S_2 \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&\geq \sum_{i \in S_1 \cup \{0\} \text{ s.t. } z_i > 0} u_i \cdot z_i - \min_{\mu_{|S_1 \cup \{0\}} \in \mathscr{W}_{S_1 \cup \{0\}}} \left[\sum_{ij \in (S_1 \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&\quad + \sum_{i \in S_2 \cup \{0\} \text{ s.t. } z_i > 0} u_i \cdot z_i - \min_{\mu_{|S_2 \cup \{0\}} \in \mathscr{W}_{S_2 \cup \{0\}}} \left[\sum_{ij \in (S_2 \cup \{0\})^2} \mu_{ij} \cdot c_{ij} \right] \\
&\geq v(S_1) + v(S_2)
\end{aligned}$$

By definition, v is supermodular. □

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