Bargaining and Buyout: A Noncooperative Bargaining Approach to Strategic Alliances

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To analyze players' strategic alliance behavior, we introduce a new noncooperative coalitional bargaining model, in which each player can *buy out* other players with upfront transfers. We uncover the role of an *essential player* in a transferable utility game, or a *veto player* in a simple game, in preventing efficient outcomes and we show that delay in bargaining generically occurs. In an application to legislative bargaining with vote buying, if a veto player and a non-veto player coexist, then a non-winning coalition forms as an intermediate bargaining step and the final winning coalition is not necessarily minimal. In three-player simple games, we fully characterize the set of the equilibrium outcomes, which is equivalent to the convex hull of the core and the egalitarian solution. As the set of equilibrium outcomes includes well-known cooperative power indices, players' strategic alliance can be viewed as a noncooperative foundation of the cooperative solutions.

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JEL Classification: C72, C78; D72, D74

1 INTRODUCTION

When three or more players bargain over their joint surplus, an intermediate subcoalition occasionally forms though it is inefficient. Rather than immediately forming an efficient coalition, players may increase their bargaining power by forming an inefficient coalition as an intermediate bargaining step. In wage bargaining, for instance, workers form a labor union even though the union itself produces nothing. Similarly, in legislative bargaining, minor parties form a coalition though the coalition is still minor. Such strategic formation of intermediate coalitions may cause delays and inefficiency, but they may also substantially alter how the players divide the surplus.

To investigate strategic alliance behaviours, we introduce a new noncooperative bargaining model in which players can form an intermediate coalition by *buying out* other players' resources and rights with upfront transfers.[1] Therefore, players strategically choose their bargaining partners, not only to obtain a surplus from a current coalition, but also to increase future collective bargaining power in subsequent games.

As an example, consider a voting game with three parties: One big party has two votes, and two small parties have one vote apiece. They are supposed to divide a dollar by supermajority rule, that is, at least three votes are needed to pass a certain division. In the existing noncooperative legislative bargaining model, for instance, Baron and Ferejohn [1989] and Winter [1996], a randomly selected proposer proposes a division of the dollar, and then all the parties vote for the proposal. Since the big party's agreement is essential to implement any division, the small parties must compete with each other to attract the big party, but they have no reason to cooperate. Due to the lack of strategic alliances in the existing model, it turns out the big party takes all the surplus in equilbrium as the discount factor approaches 1.

Alternatively, in order to consider strategic alliances, let us allow *vote buying* so that any party who obtains more than three votes wins the prize. Now small parties may want to buy out each

[1]Such transactions among players appear either explicitly or implicitly in practice, in various forms, in many societies, economies, and organizations. Examples include vote buying in legislatures, unifying candidates in elections, logrolling among political parties, and proxy voting for corporate control and takeover battles.

other's vote and form an intermediate coalition, rather than directly bargaining with the big party. Once the small parties form a coalition, although it is not a winning coalition, the coalition with two votes and the big party play a standard two-player bargaining game later, in which the small parties get a substantial amount of payoff. Therefore, the small parties not only compete but also cooperate with each other in our model. In this particular example, the equilibrium payoff in the model with strategic alliances coincides to the Shapley value, $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, as the discount factor converges to 1. (See Example 3.6 for details.)

In this paper, we extend the idea of vote buying to general transferable utility environments, which is represented by a characteristic function defined on the set of possible coalitions. The bargaining game proceeds as follows in discrete time. A proposer is randomly selected in each period. The proposer makes an offer specifying a coalition to bargain with and monetary transfers to each member in the proposed coalition. If all the members of the coalition accept the offer, then the proposer inherits other respondents' resources and rights and controls the coalition thereafter. If any of the respondents rejects the offer, then it dissolves without changing the coalitional state. At the end of each period, each remaining player derives a per-period payoff from her coalition according to the given characteristic function. We assume that all the players have a common discount factor.

Our main theorem shows that an efficient equilibrium is generically impossible due to strategic delay in forming an efficient coalition. To be specific, any stationary subgame perfect equilibrium cannot be efficient for sufficiently high discount factors if the underlying characteristic function holds the following two properties: 1) it is *not a unanimity game*, that is, there is a strict subcoalition that generates a strictly positive worth, and 2) it has an *essential player*, that is, at least one player can make a strictly positive marginal contribution to the grand coalition.

It is particularly interesting to apply the result to a class of *simple game*s, where each coalition is either all-powerful, known as a *winning coalition*, or completely ineffectual. In such games, an essential player is nothing but a *veto player*, able to prevent the formation of a winning coalition. In any simple game with at least one veto player, a non-winning coalition may form as an intermediate bargaining step. Furthermore, the final winning coalition is not necessarily minimal, unless all the players are veto players; if they are, then it is a unanimity game and no delay occurs.

In addition, we fully characterizes a set of equilibrium outcomes for three-player simple games to study the role of intermediate coalitions in splitting the joint surplus. It turns out that the set of equilibrium outcomes contains the well-known cooperative power indices, including Shapley and Shubik [1954], Banzhaf [1964], Deegan and Packel [1978], Johnston [1978], and others, which are not necessarily in the core. Thus, players' strategic alliance can be viewed as a noncooperative foundation of the cooperative solutions.

1.1 Related Literature

In most of the noncooperative coalitional bargaining models, players have at most one chance to form a coalition and forming an intermediate coalition is not allowed. For instance, Selten [1981] and Compte and Jehiel [2010] assume *one-stage property*, that is, the game is terminated right after any coalition formation. Other models, including Chatterjee et al. [1993] and Okada [1996, 2011], assume *exclusion property*; once players form a coalition, all the players in the coalition must exit the game and they are excluded from further bargaining. In legislative bargaining models such as Baron and Ferejohn [1989] and Winter [1996], players cannot form a pre-election coalition. In those standard models, players must exit the game once they agree on forming any coalition. Hence, they consider only the surplus from the coalition which they form now, since they do not take into account either their future bargaining power or the coalitional structure further induced by the coalition formation.

As players are not allowed to form intermediate coalitions in the standard coalitional bargaining models, they tend to form an efficient coalition immediately. In Chatterjee et al. [1993] and Okada [1996], the grand coalition always forms immediately for sufficiently high discount factors, if the grand coalition has the largest per-capita worth among all coalitions; and the converse is also true as showed in Proposition 5.12 in Ray and Vohra [2015]. Similarly in the legislative bargaining models, Winter [1996] shows that the game ends with no delay only with the approval of a minimum winning coalition. In our model, however, when players have buyout options, it turns out that an efficient equilibrium is generically impossible for sufficiently high discount factors.

In addition to allowing strategic alliances among players, we consider environments with multiple efficient coalitions. Due to its complexity of dealing with multiple efficient coalitions, most of the coalitional bargaining models in economics assume that a grand coalition is the unique efficient coalition. On the other hand, particularly for analyzing political situations, the legislative bargaining literature typically studies simple games, in which multiple winning coalitions are involved but no partial surplus is allowed. We provide unified results for a rich class of environments with both multiple efficient coalitions and partial surpluses.

The notion of a buyout in multi-player bargaining is introduced by Gul [1989]. In his model, however, a player cannot choose the partners to bargain with and her strategic decision is limited on splitting the joint surplus in a randomly selected bilateral meeting. As Hart and Mas-Colell [1996] pointed out, a random-meeting model does not entirely capture players' strategic behavior. We allow players to choose their bargaining partners strategically instead of imposing random meetings.

In the formal theory of political economics, as a special case of buyout options, vote buying has been widely studied to explain why non-minimal winning coalitions are prevalent in many political situations, including Banks [2000], Dal Bo [2007], Dekel et al. [2008, 2009], Groseclose and Snyder [1996]. In those models, however, only two lobbyists or parties compete to buy non-strategic voters in a specific political environment. Our model explains a general occurrence of non-minimal winning coalitions in an abstract legislative bargaining model like Baron and Ferejohn [1989] and uncover the role of veto players in strategic alliance behaviors.

In the existing noncooperative legislative bargaining models, including Baron and Ferejohn [1989], Winter [1996], Montero [2002], Morelli and Montero [2003], Montero [2006] and Montero and Vidal-Puga [2011], gradual coalition formation or pre-election coalition formation is not yet formally studied. In those models, therefore, only a minimal winning coalition immediately forms in stationary subgame perfect equilibria; while in our model, delay may occur and the final winning coalition is not necessarily minimal.

Strategic delay has been a central issue in bargaining literature. Under incomplete information, delays in equilibrium is common even in a two-player bargaining game, for instance, Myerson [1997], Abreu and Gul [2000], and Compte and Jehiel [2002] among many. Under complete information, Chatterjee et al. [1993] and Cai [2000] provided examples of a delayed equilibrium, rely on a specific characteristic function or a restrictive bargaining protocol, while our model shows that delay generically occurs. Seidmann and Winter [1998], Okada [2000], Gomes [2005], and Lee [2018] allow renegotiations or gradual coalition formation, but they limit their analysis to a grand coalition equilibrium where only the grand coalition is efficient.[2]

This paper is organized as follows. Section 2 describes a noncooperative coalitional bargaining model with buyout options. In Section 3, the main theorem is introduced and the conditions for strategic delay are explained through examples. In Section 4, we apply the result to legislative bargaining situations and characterize the set of the equilibrium outcomes for three-player simple games. Concluding remarks follow in Section 5 and the proofs appear in Appendix.

2 A MODEL

2.1 An Environment

Let N be a set of players and $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$ be a *characteristic function.* A tuple (N,v) is an *underlying characteristic function form game*, or shortly an *underlying game*.[3] We assume that (N,v) is *zero-normalized* ($v(\{i\}) = 0$ for all $i \in N$), *essential* ($v(N) > 0$), and *superadditive* ($v(S \cup S') \geq v(S) + v(S')$ for all $S,S' \subseteq N$ such that $S \cap S' = \emptyset$). An underlying game (N,v) is a *unanimity game* if $v(S) = 0$ for all $S \subsetneq N$ and $v(N) > 0$.

Define $\bar{v} \equiv \max_{S\subseteq N} v(S)$. Given (N,v), a coalition $S \subseteq N$ is *efficient* if $v(S) = \bar{v}$. Let $\mathbf{E}(N,v)$ (or simply $\mathbf{E}$ when there is no danger of confusion) be a set of *efficient coalitions.* A player $i \in N$ is *essential* if $v(S) < \bar{v}$ for any $S \subseteq N \setminus \{i\}$. Let K be a set of *essential player*s. It is easy to see that a player is essential if and only if her marginal contribution to the grand coalition is strictly positive, that is, $K = \cap\mathbf{E} = \{k \in N \mid v(N \setminus \{k\}) < \bar{v}\}$. It is easy to verify that all the players are essential if and only if the grand coalition is the unique efficient coalition, that is, $N = K \iff \mathbf{E} = \{N\}$.

A (coalitional) *state* π specifies a set of *active player*s $N^\pi \subseteq N$ and a partition $\{S_i\}_{i\in N^\pi}$ of N such that $S_i \cap N^\pi = \{i\}$, representing a distribution of resources. Abusing notation, π also refers the partition in the state. Let $[i]_\pi$ be a player i's partition block in π. Denote π° by the initial state, that is, $N^{\pi^\circ} = N$ and $[i]_{\pi^\circ} = \{i\}$ for all $i \in N$. A state π is *efficient* if $\sum_{i\in N^\pi} v([i]_\pi) = \bar{v}$. That is, in any efficient state, there is no unrealized surplus. Let Π be a set of all states.

For each $\pi \in \Pi$ and $i \in N^\pi$, denote $N^\pi_i = \{S \subseteq N^\pi \mid i \in S\}$. For each $\pi \in \Pi$, $i \in N^\pi$, and $S \subseteq N^\pi_i$, player i's *S-formation*, or (i,S) formation, yields a subsequent state $\pi(i,S)$, where $N^{\pi(i,S)} = (N^\pi \setminus S) \cup \{i\}$, $[i]_{\pi(i,S)} = \cup_{k\in S}[k]_\pi$, and $[j]_{\pi(i,S)} = [j]_\pi$ for all $j \in N^\pi \setminus S$.

When there is no danger of confusion, we omit π° in notations, for instance, $N^{\pi^\circ(i,S)} = N^{(i,S)}$. For notational simplicity, for any $z \in \mathbb{R}^n$ and $S \subseteq N$, denote $z_S = \sum_{j\in S} z_j$. For a characteristic function v, denote $v_i = v(\{i\})$ and $v_S = \sum_{i\in S} v_i$. For a finite set S, let $\Delta(S) \equiv \{p \in \mathbb{R}^{|S|} \mid p_S = 1\}$ be a simplex of S and $\Delta^\circ(S) \equiv \{p \in \Delta(S) \mid (\forall i \in S)\ p_i > 0\}$ be an interior of $\Delta(S)$.

2.2 A Noncooperative Game

For an underlying characteristic function form game (N,v), a *noncooperative coalitional bargaining game*, or shortly, a *bargaining game* is a tuple $\Gamma = (N,v,p,\delta)$, where $p \in \Delta^\circ(N)$ is the *initial recognition probability* and $0 < \delta < 1$ is the common discount factor. For each $\pi \in \Pi$, we define the

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[2] Seidmann and Winter [1998], Okada [2000], and Gomes [2005] discussed that efficient outcomes can be achieved in the two following different aspects. First, for sufficiently low discount factors no delay occurs because delay is too costly to the players. Second, as the discount factor converges to unity an equilibrium is asymptotically efficient because inefficiency caused by delay becomes negligible. In an environment with a unique efficient coalition, our findings are consistent with those existing results, but we provide a clear condition for inefficiency. More importantly, we extend the results to environments with multiple efficient coalitions and find the role of an essential player in strategy delay.

[3] We follows Gul [1989]'s interpretation. Each player initially has a specific resource. Each coalition represents a combination of resources that initially belong to the players in the coalition which generates a flow of surplus according to the characteristic function.

induced recognition probability $p^\pi \in \Delta^\circ(N^\pi)$ as $p_i^\pi = p_{[i]_\pi}$ for all $i \in N^\pi$. That is, if a player forms a coalition, then the player takes other players' recognition probabilities as well.

A bargaining game proceeds as follows. In each period $t = 1, 2, \cdots$, it begins with the previous state π^{t-1}. If π^{t-1} is an efficient state, then only a production stage occurs without bargaining stages and hence $\pi^t = \pi^{t-1}$. Otherwise, the period consists of three bargaining stages and one production stage. Each stage is defined as follows:

(1) *Recognition:* Nature selects a player $i \in N^{\pi^{t-1}}$ as a proposer with probability $p_i^{\pi^{t-1}}$.
(2) *Proposal:* The proposer i makes an offer by choosing a pair (S, y) of a coalition $S \subseteq N_i^{\pi^{t-1}}$ and monetary transfers $\{y_j\}_{j \in S}$.
(3) *Response:* By a given order, each respondent $j \in S \setminus \{i\}$ sequentially either accepts the offer or rejects it. If any $j \in S \setminus \{i\}$ rejects then the current state does not change and hence $\pi^t = \pi^{t-1}$. If all $j \in S \setminus \{i\}$ accept the offer, then the current state transitions to $\pi^t = \pi^{t-1}(i, S)$, that is, each $j \in S \setminus \{i\}$ leaves the game with receiving y_j from the proposer i.
(4) *Production:* Each partition block generates a surplus to the owner. That is, each active player $i \in N^{\pi^t}$ derives $(1 - \delta)v([i]_{\pi^t})$.[4]

2.3 Stationary Subgame Perfect Equilibria

Our equilibrium concept is a *stationary subgame perfect equilibrium*, or an *SSPE* in short. In this subsection, we introduce the equilibrium concept and provide a couple of preliminary results, Proposition 2.1 and Proposition 2.2, of which proofs can be found in the literature including Yan [2003], Eraslan and McLennan [2013] and Lee [2014].

A player's strategy is *stationary* if it does not depend on the histories of past periods. Note that even in the class of stationary strategies, players' decision may depend on the current state and *within-period* histories, which involve the identity of the proposer, the proposed coalition, preceding respondents' reactions, and so on.

An SSPE consists of each player's stationary strategy, which is the player's best response to other players' strategies in every subgames. It is worth mentioning that in an SSPE each player has no incentive to play any other strategies including even non-stationary strategies, as long as other players are supposed to play stationary strategies.

The notion of SSPE is widely adopted as a *focal* equilibrium in the coalitional bargaining literature. Unlike in two-player noncooperative bargaining models, it is well-known that any feasible allocation can be achieved by a subgame perfect equilibrium for high discount factors in most of the multi-player bargaining models.[5] Without the notion of stationarity, therefore, a noncooperative multi-player bargaining model usually fails to provide a sharp prediction.

To analyze SSPE, we introduce a special form of stationary strategies, so-called a *cutoff strategy*. Before defining a cutoff strategy and a cutoff strategy equilibrium, we state Proposition 2.1, which provides a payoff equivalence result between a cutoff strategy equilibrium and a general SSPE. Due to this result, we may focus on cutoff strategy equilibria instead of all SSPE without loss of generality, when we are interested in either players' equilibrium payoffs or efficiency.

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[4]The coefficient $1 - \delta$ normalizes the discounted sum of streams of surplus. Thus, a coalition S generates $(1 - \delta)v(S)$ for each period so that the sum of streams of surplus is $v(S) = (1 - \delta)v(S) + \delta(1 - \delta)v(S) + \delta^2(1 - \delta)v(S) + \cdots$.

[5]Shaked (reported by Sutton [1986]) for a multi-lateral unanimity game, Baron and Ferejohn [1989] for a legislative bargaining, and Chatterjee et al. [1993] for a transferable utility game. A notable exception is Krishna and Serrano [1996], who allow intermediate coalition formation to establish a multi-lateral unanimity game with unique subgame perfect equilibrium without stationarity. Proposition 3.1 overlaps with their result.

Proposition 2.1. *For any SSPE, there exists a cutoff strategy equilibrium which yields the same payoff vector.*

A cutoff strategy profile $(\mathbf{x},\mathbf{q})$ consists of a cutoff value profile $\mathbf{x} = \{\{x_i^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi}$ and a coalition formation strategy profile $\mathbf{q} = \{\{q_i^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi}$, where $x_i^\pi \in \mathbb{R}$ and $q_i^\pi \in \Delta(2^{N_i^\pi})$ for each $\pi \in \Pi$ and it specifies the behaviors of each active player $i \in N^\pi$ in any coalitional state π in the following way: 1) whenever player i becomes a proposer to propose (S,y), she chooses a coalition S with probability $q_i^\pi(S)$ and offers $y_k = x_k^\pi$ to each $k \in S\setminus\{i\}$ and 2) whenever player i becomes a respondent, she accepts any proposal (S,y) if and only if $y_i \geq x_i^\pi$.

Given $\mathbf{x}$, define an active player i's *excess surplus* of forming S in π as the difference between her cutoff value in the subsequent state and the sum of cutoff values of the members in the coalition S, that is,

$$e_i^\pi(S,\mathbf{x}) = x_i^{\pi(i,S)} - x_S^\pi.$$

When there is no danger of confusion, we omit the subscript on e, as $e_i(S) = e_j(S)$ for all $S \subseteq N$ such that $|S| \geq 2$ and all $i,j \in S$. Given a cutoff strategy profile $(\mathbf{x},\mathbf{q})$, player i's *continuation payoff* in π is:

$$u_i^\pi(\mathbf{x},\mathbf{q}) = p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) e_i^\pi(S,\mathbf{x}) + \sum_{j\in N^\pi} p_j^\pi \sum_{S\subseteq N^\pi} q_j^\pi(S)\left[\mathbb{1}(i\in S)x_i^\pi + \mathbb{1}(i\notin S)x_i^{\pi(j,S)}\right]. \quad (1)$$

The main benefit of using a cutoff strategy equilibrium is its tractability. Proposition 2.2 characterizes a cutoff strategy equilibrium in terms of a value profile and a coalition formation strategy profile.

Proposition 2.2. *A cutoff strategy profile* $(\mathbf{x},\mathbf{q})$ *is an SSPE if and only if for all* $\pi \in \Pi$ *and* $i \in N^\pi$,

i) **Optimality:** *player i chooses a coalition to maximizes her excess surplus, that is,*

$$q_i^\pi(S) > 0 \implies (\forall S' \subseteq N_i^\pi)\ e_i^\pi(S,\mathbf{x}) \geq e_i^\pi(S',\mathbf{x}); \quad \textit{and}$$

ii) **Consistency:** *player i's cutoff value is consistent with the sum of the current per-period surplus and the discounted continuation payoff, that is,*

$$x_i^\pi = (1-\delta)v([i]_\pi) + \delta u_i^\pi(\mathbf{x},\mathbf{q}).$$

Given $\Gamma = (N,v,p,\delta)$, a cutoff strategy profile $(\mathbf{x},\mathbf{q})$ is *efficient* if $\sum_{i\in N} u_i(\mathbf{x},\mathbf{q}) = \bar{v}$. Since any proposal is always accepted under a cutoff strategy profile, a cutoff strategy profile $(\mathbf{x},\mathbf{q})$ is efficient if and only if, for all $i \in N$ and $S \subseteq N$, $q_i(S) > 0$ implies $v(S) = \bar{v}$.

3 THE MAIN RESULT: CONDITIONS FOR INEFFICIENCY

Before stating the main result, we review some of the preliminary results about efficiency. In a unanimity game, first of all, it is well known that any subgame perfect equilibrium is efficient and the equilibrium payoff vector is unique.[6] In fact, players in a unanimity game have no incentive to form a strict subcoalition as an intermediate bargaining step as only the grand coalition generates a positive surplus. Proposition 3.1 re-states the well-known result within our setting. In a unanimity game, forming a strict subcoalition must yield another (but smaller) unanimity game. Due to this property, Proposition 3.1 can be proved by induction.

Proposition 3.1 (Unanimity Game). *Suppose* (N,v) *is a unanimity game. For any* p *and* δ, *any stationary equilibrium of* (N,v,p,δ) *is efficient and the equilibrium payoff vector is* $\bar{v}p$.

[6]Note that the efficiency and the uniqueness of equilibrium hold even without imposing stationarity on strategies. Krishna and Serrano [1996] consider intermediate bargaining steps to obtain a unique equilibrium of a multilateral bargaining game in which only a unanimous agreement generates a (possibly non-transferable) surplus.

The coalitional bargaining literature in economics tends to focus on a grand coalition equilibrium, in which all the players always immediately form a grand coalition. Other than in a unanimity game, it turns out a grand coalition equilibrium is impossible as Proposition 3.2 shows.[7] It can be proved by contradiction: if a grand coalition equilibrium is assumed, then at least one player can be strictly better off by forming a strict subcoalition.

Proposition 3.2 (Grand Coalition Equilibria). *Suppose (N, v) is not a unanimity game. For any p, there exists $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, a bargaining game (N, v, p, δ) has no grand coalition equilibrium.*

As a direct consequence of Proposition 3.1 and Proposition 3.2, Corollary 3.3 highlights the impact of allowing strategic alliances. Recall that, in the standard models without buyout options such as Chatterjee et al. [1993], Okada [1996], and Compte and Jehiel [2010], a grand coalition always immediately forms if the grand coalition has the largest per-capita worth, that is, $\frac{v(N)}{|N|} \geq \frac{v(S)}{|S|}$ for all $S \subseteq N$. With buyout options, however, a grand coalition equilibrium is impossible for high discount factors if there exists $S \subsetneq N$ such that $v(S) > 0$.

Corollary 3.3. *Consider any underlying game (N, v) and any recognition probability p. A bargaining game (N, v, p, δ) has a grand coalition equilibrium for all δ if and only if (N, v) is a unanimity game.*

While a grand coalition equilibrium is of course efficient for a superadditive game, the converse is not true as a strict subcoalition may also be efficient. When multiple efficient coalitions are involved, characterizing an efficient equilibrium is challenging because the different efficient coalitional states should be all considered. The main theorem extends the impossibility result of grand coalition equilibrium to characterize conditions for an inefficient equilibrium.

Theorem 3.4. *For any non-unanimity game (N, v) with an essential player and any recognition probability p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ the noncooperative game (N, v, p, δ) has no efficient stationary subgame perfect equilibrium.*

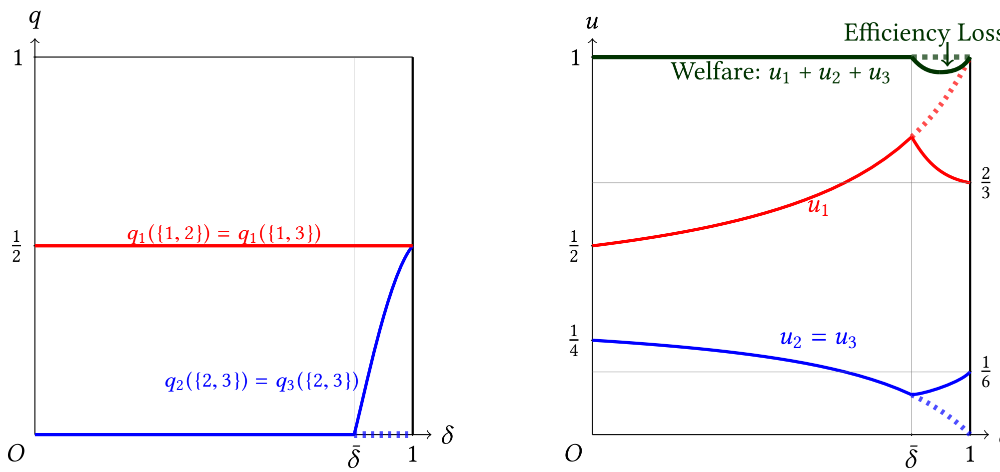
To prove Theorem 3.4, we first generalize Winter [1996]'s result on a simple game with a veto player to a general characteristic function form game with an essential player. In any efficient equilibrium, Lemma 3.5 says that essential players must take all the surplus as a discount factor converges to 1.

Lemma 3.5 (Generalized Winter's Theorem). *Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium of (N, v, p, δ). If $K \neq \emptyset$, then x_K converges to $\bar{v}$ as δ approaches 1.*

Using Lemma 3.5, we prove Theorem 3.4 by contradiction. When players have buyout options for strategic alliances, even a non-essential player gets a strictly positive payoff, which contradicts to Generalized Winter's Theorem and hence it turns out an efficient equilibrium is impossible. The only case in which essential players take all the surplus is when all the players are essential as Proposition 3.2 shows.

The proof considers three possible cases to find a contradiction and each case provides a different insight for strategic delay. In the first case, if $K \notin \mathbf{E}$, that is, the set of essential players is not efficient, then non-essential players form a coalition among themselves to become a new collective essential player against the current essential players, instead of directly forming an efficient coalition with

[7] This compliments the existing result by Seidmann and Winter [1998], Okada [2000], and Gomes [2005]. They discussed that efficient outcomes without delay can be achieved for sufficiently low discount factors because delay is too costly to the players.



(a) *Coalition Formation Strategies:* For lower discount factors ($\delta \leq \bar{\delta}$), the small players never form a coalition $\{2, 3\}$ with each other but they always choose the big player as a bargaining partner. However, as a discount factor increases over the threshold $\bar{\delta}$, the small players form $\{2, 3\}$ with a positive probability $0 < q_2(\{2, 3\}) = q_3(\{2, 3\}) < \frac{1}{2}$ rather than immediately forming a winning coalition. That is, the small players not only compete but also cooperate with each other. On the other hand, when players have no buyout options, $\{2, 3\}$ never forms for any discount factor, as the dotted line depicts that $q_2(\{2, 3\}) = q_3(\{2, 3\}) = 0$.

(b) *Equilibrium Payoffs:* When players have no buyout option, the big player takes all the surplus as $\delta \to 1$ (See the dotted lines). When players have buyout options and $\delta > \bar{\delta}$, the small players' payoff increases as δ increases because they form an intermediate coalition $\{2, 3\}$ more frequently. In the limit of $\delta \to 1$, the equilibrium payoff vector coincides with the Shapley value $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ in this particular example. Inefficiency occurs for any $\delta \in (\bar{\delta}, 1)$ as forming a winning coalition is delayed. However, it is asymptotically efficient; as $\delta \to 1$, delay becomes less costly though it occurs more frequently.

Fig. 1. The Role of High Discount Factors in Three-Party Weighted Majority Game (Example 3.6)

the essential players. In the second case, if $K \in \mathbf{E}$ and there exists an essential player $k' \in K$ such that $v(N \setminus \{k'\}) > 0$, then there exists another essential player $k \in K$ who can be better off by forming an inefficient coalition excluding k'. In the last case, if $K \in \mathbf{E}$ and $v(N \setminus \{k'\}) = 0$ for all for $k' \in K$, then essential players compete with each other to buy out non-essential players rather than immediately forming an efficient coalition. The complete proof is shown in Appendix.

Theorem 3.4 characterizes the three necessary conditions for strategic delay. That is, a strategic delay occurs with positive probability and hence an efficient equilibrium is impossible, if 1) the discount factor is high enough, 2) at least one strict subcoalition generates a positive worth, and 3) at least one player is essential. We illustrate the role of each condition with examples in the following three subsections in turn.

3.1 High Discount Factors

If the discount factor is sufficiently low, then players do not care about their future payoffs. Thus, any respondent immediately accepts any positive offer, and hence any proposer chooses an efficient coalition to extract most of the surplus from the efficient coalition. In other words, players have no incentive to form an intermediate coalition if the discount factor is lower than a certain level. If the discount factor is greater than the threshold, however, players may form an inefficient coalition as an intermediate step, expecting more bargaining power in the subsequent bargaining, and hence a strategic delay occurs. The following example illustrates the role of a discount factor in efficiency.

Example 3.6 (A Three-Party Weighted Majority Game). Suppose there are three players: one big player has two votes and each of the other two small players has one vote. They split a dollar by a supermajority rule, that is, at least 3 votes are required to pass a division of the dollar. Assume that their recognition probabilities are proportional to their voting weights. Then the corresponding noncooperative bargaining game consists of $N = \{1,2,3\}$, $v(S) = 1$ if $S \in \{\{1,2\},\{1,3\},N\}$ and $v(S) = 0$ otherwise, and $p = \left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$. In any subgame with two active players, the equilibrium strategy is uniquely determined in a standard way, and hence $x_2^{(2,\{2,3\})} = x_3^{(3,\{2,3\})} = (p_2+p_3)\delta$ and $x_1^{(2,\{2,3\})} = x_1^{(3,\{2,3\})} = p_1\delta$. Now we consider the strategies in the initial state.

There are two types of equilibria depending on δ. Let $\bar{\delta} = \frac{6-2\sqrt{3}}{3} \approx 0.845$.

(1) **(Efficient Equilibrium)** If $\delta \le \bar{\delta}$, either $\{1,2\}$ or $\{1,3\}$ always immediately forms. There exists an equilibrium which consists of
- $x = \left(\frac{(2-\delta)\delta}{4-3\delta}, \frac{(1-\delta)\delta}{4-3\delta}, \frac{(1-\delta)\delta}{4-3\delta}\right)$;
- $q_1(\{1,2\}) = q_1(\{1,3\}) = \frac{1}{2}$; and
- $q_2(\{1,2\}) = q_3(\{1,3\}) = 1$.

To see the proposed strategies construct an equilibrium, we verify the two conditions in Proposition 2.2, optimality and consistency. To verify optimality, first we observe:

$$e(\{1,2\}) \ge e(\{2,3\}) \iff 1 - \frac{(3-2\delta)\delta}{4-3\delta} \ge \frac{1}{2}\delta - \frac{2(1-\delta)\delta}{4-3\delta} \iff \delta \le \bar{\delta}.$$

Furthermore, $e(N) = 1 - x_N = 1 - \delta$ is strictly less than than $e(\{1,2\}) = e(\{1,3\}) = 1 - x_1 - x_2 = 1 - \frac{(3-2\delta)\delta}{4-3\delta}$. Thus, we have $\operatorname{argmax}_S e(S) = \{\{1,2\},\{1,3\}\}$ and optimality holds. To verify consistency, compute each player's expected payoff:

$$\begin{aligned} u_1 &= p_1\left[q_1(\{1,2\})e(\{1,2\}) + q_1(\{1,3\})e(\{1,3\})\right] + (p_1+p_2+p_3)x_1 \\ &= \tfrac{1}{2}\left(1 - \tfrac{(3-2\delta)\delta}{4-3\delta}\right) + 1 \cdot \tfrac{(2-\delta)\delta}{4-3\delta} = \tfrac{2-\delta}{4-3\delta}, \\ u_2 &= p_2 e(\{1,2\}) + (p_2 + p_1 q_1(\{1,2\}))x_2 \\ &= \tfrac{1}{4}\left(1 - \tfrac{(3-2\delta)\delta}{4-3\delta}\right) + \tfrac{1}{4} \cdot \tfrac{(1-\delta)\delta}{4-3\delta} = \tfrac{1-\delta}{4-3\delta}. \end{aligned}$$

Thus, $x_i = \delta u_i$ for all $i \in N$, and hence consistency holds.

(2) **(Inefficient Equilibrium)** If $\delta > \bar{\delta}$, then the small players form an intermediate coalition $\{2,3\}$ with positive probability. There exists an equilibrium with
- $x = \left(1 - \frac{\delta+2\bar{e}}{4}, \frac{\delta-2\bar{e}}{4}, \frac{\delta-2\bar{e}}{4}\right)$;
- $q_1(\{1,2\}) = q_1(\{1,3\}) = \frac{1}{2}$;
- $q_2(\{2,3\}) = 1 - q_2(\{1,2\}) = q_3(\{2,3\}) = 1 - q_3(\{1,3\}) = r > 0$,

where $\bar{e} = 1 - \frac{2}{3}\delta - \frac{1}{6}\sqrt{-12\delta^3 + 61\delta^2 - 84\delta + 36}$ and $r = \frac{2(2\delta-\delta^2-4\bar{e})}{\delta(\delta-2\bar{e})}$. Note that $\bar{e}$ is the solution to $\frac{2\delta-\delta^2-4\bar{e}}{\delta-2\bar{e}} = \frac{5\delta-\delta^2-4+2\bar{e}}{4-3\delta-2\bar{e}}$ and hence

$$r = \frac{2(2\delta-\delta^2-4\bar{e})}{\delta(\delta-2\bar{e})} = \frac{2(5\delta-\delta^2-4+2\bar{e})}{\delta(4-3\delta-2\bar{e})}$$

Now we verify the strategy profile constitutes an equilibrium. First, given x, it is easy to see $e(\{1,2\}) = e(\{1,3\}) = e(\{2,3\}) = \bar{e} > e(N) = 1-\delta$. Furthermore, observe that

$$0 < r < 1 \iff \frac{2\delta-\delta^2}{4} < \bar{e} < \frac{4\delta-3\delta^2}{8-2\delta} \iff \bar{\delta} < \delta < 1,$$

which implies optimality. Thus each proposer obtains an excess surplus of $\bar{e}$ and each player's expected payoff is:

$$\begin{aligned}
u_1 =& p_1\bar{e} + [p_1 + p_2q_2(\{1,2\}) + p_3q_3(\{1,3\}))]x_1 + [p_2q_2(\{2,3\} + p_3q_3(\{2,3\})]\delta p_1 \\
=& \tfrac{1}{2}\bar{e} + x_1 - \tfrac{1}{2}r\left(x_1 - \tfrac{\delta}{2}\right) \\
=& 1 - \tfrac{\delta}{4} - \tfrac{1}{2}\left(\tfrac{2(5\delta-\delta^2-4+2\bar{e})}{\delta(4-3\delta-2\bar{e})}\right)\left(\tfrac{4-3\delta-2\bar{e}}{4}\right) \\
=& \tfrac{1}{\delta}\left(1 - \tfrac{\delta+2\bar{e}}{4}\right) = \tfrac{1}{\delta}x_1 \\
u_2 =& p_2\bar{e} + [p_1q_1(\{1,2\}) + p_2 + p_3q_3(\{2,3\})]x_2 + [p_1q_1(\{1,3\}) + p_3q_3(\{1,3\})] \cdot 0 \\
=& \tfrac{1}{4}\bar{e} + \tfrac{1}{2}x_2 + \tfrac{1}{4}rx_2 \\
=& \tfrac{\delta}{8} + \tfrac{1}{4}\left(\tfrac{2(2\delta-\delta^2-4\bar{e})}{\delta(\delta-2\bar{e})}\right)\left(\tfrac{\delta-2\bar{e}}{4}\right) \\
=& \tfrac{1}{\delta}\left(\tfrac{\delta-2\bar{e}}{4}\right) = \tfrac{1}{\delta}x_2,
\end{aligned}$$

which confirms consistency. Note that $\lim_{\delta\to1}\bar{e} = \frac{1}{6}$, $\lim_{\delta\to1} r = 1$, and $\lim_{\delta\to1} x = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, which coincides with the Shapley value of the underlying game.

Figure 1 depicts the equilibrium payoffs and coalition formation strategies in the game and presents the role of the buyout option. □

3.2 Non-Unanimity

The following example illustrates the role of a strict subcoalition that generates a positive worth. Such a strict coalition works as a fulcrum for a player to form an intermediate coalition instead of immediately forming an efficient one. However, if only the grand coalition generates a positive worth, that is, the underlying game is a unanimity game, then all the players always immediately form a grand coalition.

Example 3.7 (An Employer-Employee Game). Consider a three-player game with $N = \{1,2,3\}$. Suppose $v(N) = 1$, $v(\{1,2\}) = w_2$, and $v(\{1,3\}) = w_3$ with $0 \le w_2, w_3 < 1$; and all other coalitions generate zero. Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium so that the grand coalition always forms immediately. Hence each player i's expected payoff is $u_i(\mathbf{x}, \mathbf{q}) = p_i(1 - x_N) + x_i$. Since it is efficient, it must be $u_N(\mathbf{x}, \mathbf{q}) = 1$. Due to consistency, multiplying by δ yields $x_i = \delta p_i(1 - \delta) + \delta x_i$, and hence $x_i = \delta p_i$. Take $j \in \{2,3\}$ and let $k = \{2,3\} \setminus \{j\}$. The excess surplus from forming $\{1,j\}$ is:

$$\begin{aligned}
e(\{1,j\}) &= x_1^{(1,\{1,j\})} - (x_1 + x_j) = (w_j + \delta(p_1 + p_j)(1 - w_j)) - \delta(p_1 + p_j) \\
&= ((1-\delta) + \delta p_k)w_j.
\end{aligned}$$

Player 1's optimality implies that $e(N) \ge e(\{1,j\})$, or equivalently, $\delta \le \frac{1-w_j}{1-w_j+p_kw_j}$. If $p_kw_j > 0$, then $\frac{1-w_j}{1-w_j+p_kw_j} < 1$, and hence the optimality is violated for any $\delta > \frac{1-w_j}{1-w_j+p_kw_j}$. Therefore, in order for an efficient strategy profile $(\mathbf{x}, \mathbf{q})$ to be an equilibrium, it must be $p_2w_3 = p_3w_2 = 0$. As long as $p_2 > 0$ and $p_3 > 0$, an efficient equilibrium is impossible for a sufficiently high discount factor, unless $w_2 = w_3 = 0$. If $w_2 = w_3 = 0$, then (N, v) is a unanimity game and the efficient strategy profile constructs an equilibrium for any discount factor. □

3.3 Essential Players

Lastly, we investigate the role of essential players. A trivial example with no essential player is a three-player simple majority game. In this case with no essential player, any two-player coalition is a winning coalition and hence there is no room for an intermediate coalition formation. A simplest non-trivial example with no essential player is an apex game.

Example 3.8. [An Apex Game] Consider a four-player apex game: one apex player has two votes and each of the other three minor players has one vote; and at least three votes are needed to win. Note that as the three minor players can form a winning coalition, no player has veto power and there is no essential player. However, if any two minor players form a coalition, then it becomes a *collective veto coalition* in the subsequent game. Thus, our concern is whether any pair of minor players has an incentive to form a collective veto coalition instead of immediately forming a winning coalition.
Assume that their recognition probabilities are proportional to their voting weights, $p = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, and the discount factor is close to 1. If all the players immediately form a winning coalition, then the equilibrium payoff vector should be $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, according to Montero [2002]. If a minor player forms a veto coalition with another minor player, then the two minor players can expect $\frac{7}{15}$ jointly in the next period, based on Proposition 4.5. Thus, the excess surplus of forming a veto coalition is then $\frac{7}{15} - 2 \times \frac{1}{5} = \frac{1}{15}$, which is less than that of forming a winning coalition, $1 - 3 \times \frac{1}{5} = \frac{2}{5}$. Therefore, any minor player has no incentive to form a veto coalition as an intermediate bargaining step as long as the other players stick on efficient strategies, and this confirms that the efficient strategy profile constitutes an equilibrium. This argument is, of course, true for lower discount factors in which the players discount more on future payoffs. Therefore, for any discount factor, all the players always immediately form a winning coalition. □

In an apex game, if minor players form a collective veto coalition, the apex player also becomes a veto player in the subsequent game. Such a positive externality of forming a veto coalition may hinder inefficiency with intermediate coalition formation. However, one can find other examples for an efficient equilibrium with no veto player, in which some players can form a unique veto coalition.[8] The underlying driving force that prevents intermediate coalitions is again the possibility of strategic alliances of the other opponents. If some players form a unique veto coalition, other remaining players may form another veto player later and hence the payoff of the collective veto coalition is limited. Thus, a player may want to form a winning coalition immediately after being selected as a proposer. When veto players (or essential players) already exist, however, players may have a stronger incentive to form an intermediate coalition against the veto players.

4 APPLICATIONS: LEGISLATIVE BARGAINING WITH VOTE BUYING

In the class of simple games, our model can be interpreted as legislative bargaining with vote buying. In this section, we re-state the main results as corollaries for simple games and highlight the role of a veto player in inefficiency. We then investigate the impact of strategic alliances on inequality by characterizing the set of equilibrium outcomes in three-player simple games and comparing it to the well-known cooperative solution concepts.

A simple game is formally defined in the following way. Given a set of players N, a class of subsets $\mathbf{W} \subset 2^N$ is a set of (proper) *winning coalitions* if $(\forall i \in N)\ \{i\} \notin \mathbf{W}$; $N \in \mathbf{W}$; $S \in \mathbf{W} \implies (\forall S' \supset S)\ S' \in \mathbf{W}$; and $S \in \mathbf{W} \implies (N \setminus S) \notin \mathbf{W}$. A characteristic function form game (N, v) is a (proper) *simple game* if $v(S) = 1$ for all $S \in \mathbf{W}$ and $v(S) = 0$ otherwise. Let

[8]Consider a four-player weighted voting game, in which player 1 has three votes, each of player 2 and player 3 has two votes, player 4 has one vote, and at least five votes are needed to win. Note that player 2 and player 3 can form a unique veto coalition, but one can construct an efficient equilibrium for any discount factor in which they always immediately form a winning coalition.

$\mathbf{W}^m = \{S \in \mathbf{W} \mid (\forall i \in S)\ S \setminus \{i\} \notin \mathbf{W}\}$ be a set of *minimal winning coalition*s, and $V = \cap \mathbf{W}$ a set of *veto player*s.

For a simple game with a veto player, a non-winning coalition forms with positive probability as an intermediate bargaining step for sufficiently high discount factors, unless all the players are veto. In this case, the bargaining game ends up with a non-minimal winning coalition and the equilibrium payoff vector is not necessarily in the core. This is in contrast to Baron and Ferejohn [1989] and Winter [1996], in which vote buying is not allowed.

Corollary 4.1. *Let (N,v) be a simple game with a veto player. A bargaining game (N,v,p,δ) has an efficient equilibrium for all discount factors if and only if all the players are veto.*

Corollary 4.2. *Let (N,v) be a simple game with a veto player but not a unanimity game. In any equilibrium of (N,v,p,δ), there exists $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, a non-winning coalition forms with positive probability as a transitional state and the final winning coalition is not necessarily minimal.*

Now we define a set of the equilibrium payoff vectors in order to compare the equilibrium outcome in the noncooperative bargaining game and the various solutions in the underlying cooperative game. Note that the noncooperative bargaining game relies on the specific protocol which is exogenously given. In particular, the recognition probability which captures the player's implicit bargaining power is not based on the underlying cooperative game but exogenously given. Hence we want to distinguish the implicit bargaining power among players from the institution or the underlying cooperative bargaining game and we consider all the possible relative powers among players. We also concentrate on the limit case of $\delta \to 1$ to analyze an ideal environment with no friction. For any characteristic function form game (N,v), let $\mathcal{E}(N,v)$ be a set of equilibrium outcomes,

$$\mathcal{E}(N,v) := cl\left(\lim_{\delta\to 1} \bigcup_{p\in\Delta^\circ(N)} \{x \mid x \text{ constitutes an equilibrium } (\mathbf{x},\mathbf{q}) \text{ for } (N,v,p,\delta)\}\right).$$

Note that we take the closure of the set of equilibrium outcomes as recognition probabilities are chosen from the interior of the simplex.

To illustrate the impact of allowing strategic alliances in allocation, we focus on three-player simple games with $N = \{1,2,3\}$. For any $\pi \in \Pi$ such that $|N^\pi| = 2$, there exists a unique cutoff strategy equilibrium with $x_i^\pi = \delta p_i^\pi$ and $q_i^\pi(N^\pi) = 1$ for all $i \in N^\pi$. Thus, specifying strategies (x,q) in the initial state is enough for stationary subgame perfect equilibria of three-player games. We characterize an equilibrium payoff vector for the cases depend on the number of veto players. It is remarkable that the equilibrium payoff vector is uniquely determined in the limit that the discount factor converges to 1, without imposing any symmetric assumption on players' strategies.

4.1 No Veto Player

Suppose $v(S) = 1$ if $S \in \mathbf{W} = \{\{1,2\},\{1,3\},\{2,3\},N\}$ and $v(S) = 0$ otherwise, that is, any two-player coalition is a winning coalition and there is no veto player. The following proposition characterizes the equilibrium payoff vector in the case of $\delta \to 1$.

Proposition 4.3. *Let (x,q) be an equilibrium of a three-player simple game with no veto player. For any p with $p_1 \geq p_2 \geq p_3$, as $\delta \to 1$:*

i) If $p_1 > \frac{1}{2}$, then $x = \left(\frac{p_1}{2-p_1}, \frac{1-p_1}{2-p_1}, \frac{1-p_1}{2-p_1}\right)$;
ii) If $p_1 \leq \frac{1}{2}$, then $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

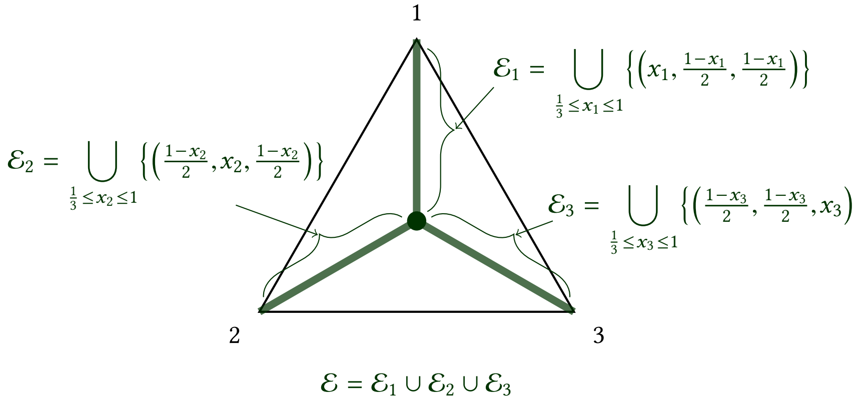


Fig. 2. $\mathcal{E}(N, v)$ with no veto player

If each player's recognition probability is less than or equal to a half, then the players' expected payoffs are all the same. However, if one of the players has a recognition probability that is greater than a half, then the dominating player gets more than $\frac{1}{3}$ and the other two players share the remaining part equally.

Figure 2 presents the set of equilibrium outcomes. If each player's recognition probability is less than or equal to a half, then the egalitarian point is the outcome as long as all the players have at least a positive chance of being a proposer. If one of the players has a recognition probability that is greater than a half, then the dominating player gets more than $\frac{1}{3}$ and the other two players share the remaining part equally.

It is not surprising that the most of the cooperative solutions allocate the egalitarian focal point in this case. However, Proposition 4.3 suggests that the egalitarian allocation might be vulnerable when one of the players has much higher bargaining power than the others even though the institution itself is symmetric.

4.2 Single Veto Player

If there is only one veto player, then the other non-veto players' payoffs are always the same, no matter what their recognition probabilities are. Furthermore, if the veto player's recognition probability is greater than $\frac{1}{2}$, then the other non-veto players will always form a coalition with each other. Suppose $v(S) = 1$ if $S \in \mathbf{W} = \{\{1,2\}, \{1,3\}, N\}$ and $v(S) = 0$ otherwise.

Proposition 4.4. *Let (x, q) be an equilibrium of a three-player simple game with $V = \{1\}$. For any p, as $\delta \to 1$:*

i) If $p_1 \geq \frac{1}{2}$, then $q_2(\{2,3\}) = q_3(\{2,3\}) = 1$ and

$$x_1 = \frac{p_1(3 - 2p_1)}{2 - p_1}, \quad \textit{and} \quad x_2 = x_3 = \frac{(1 - p_1)^2}{2 - p_1}. \tag{2}$$

ii) If $p_1 < \frac{1}{2}$, then $0 < q_2(\{2,3\}) < 1$ and $0 < q_3(\{2,3\}) < 1$, and

$$x_1 = \frac{1 + 2p_1}{3}, \quad \textit{and} \quad x_2 = x_3 = \frac{1 - p_1}{3}. \tag{3}$$

Figure 3 presents the set of equilibrium outcomes, which is the convex hull of the unique core allocation $(1, 0, 0)$ and the egalitarian allocation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The outcome depends only on the veto player's recognition probability and the two non-veto players payoffs are the same no matter what their recognition probabilities are. The well-known cooperative power indices are all in the set

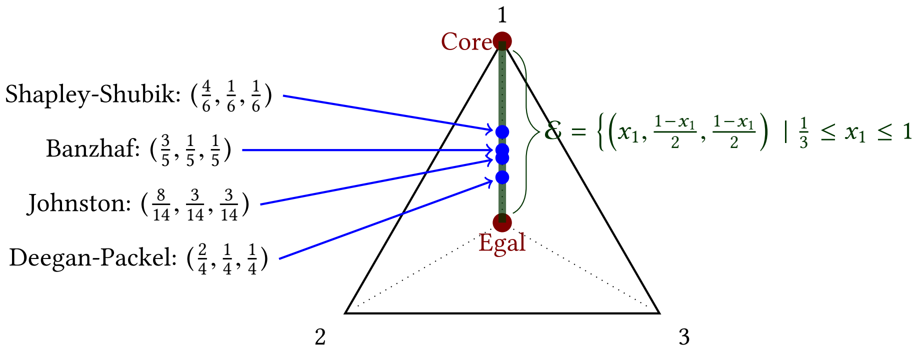


Fig. 3. $\mathcal{E}(N,v)$ with $V = \{1\}$

The set of equilibrium outcomes is the convex hull of the core and the egalitarian point. The veto player gets more than $\frac{1}{3}$ and the other two players share the remaining part equally no matter what the recognition probabilities are. The set contains well-known power indices such as Shapley-Shubik, Banzhaf, Johnston, and Deegan-Packel index.

of equilibrium outcomes: for instance, the Shapley-Shubik index $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$, Banzhaf index $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, Johnston index $(\frac{8}{14}, \frac{3}{14}, \frac{3}{14})$, and Deegan-Packel index $(\frac{2}{4}, \frac{1}{4}, \frac{1}{4})$.

4.3 Two Veto Players

Now consider the case with two veto players and one dummy player. That is, $v(S) = 1$ if $S \in \mathbf{W} = \{\{1,2\}, N\}$ and $v(S) = 0$ otherwise. Since even the dummy player has a positive recognition probability, the veto players may compete to buy out the dummy player's chance of making a proposal.

Proposition 4.5. *Let (x,q) be an equilibrium of a three-player simple game with $V = \{1,2\}$. For any p, as $\delta \to 1$, we have $q_1(\{1,3\}) > 0$ and $q_2(\{2,3\}) > 0$, and*

$$x_1 = p_1 + \frac{p_3}{3}, \qquad x_2 = p_2 + \frac{p_3}{3}, \text{ and } \qquad x_3 = \frac{p_3}{3}.$$

Figure 4 depicts the set of equilibrium outcomes. Note that the dummy player receives nothing in the core; while $\mathcal{E}$ is the convex hull of the core and the egalitarian point as in the single-veto case. However, no matter how large the dummy player's recognition probability is, each veto player gets more than the dummy player. While many cooperative solution concepts hold *dummy player property*, which requires a dummy player gets nothing, following Shapley [1952], alternative one-point solution concepts have been proposed without dummy-player property. Among many, Nowak and Radzik [1994] proposes a *solidarity value* with *average-dummy player property* and Lee and Driessen [2012] introduces a *sequentially two-leveled egalitarianism* with *scale-dummy player property*. Those cooperative solutions without dummy player property assign a non-core allocation, but belong to the set of the equilibrium outcomes.

4.4 A Unanimity Game

The last case is a unanimity game where all the players are veto. Due to Proposition 3.1, the equilibrium payoff vector is equivalent to the initial recognition probability no matter what the discount factor is. Therefore if $v(N) = 1$ and $v(S) = 0$ for any $S \subsetneq N$, then $\mathcal{E}(N,v) = Core(N,v) = \Delta(N)$. That is, the core is equivalent to the set of imputations and any core allocation can be supported as an equilibrium outcome.

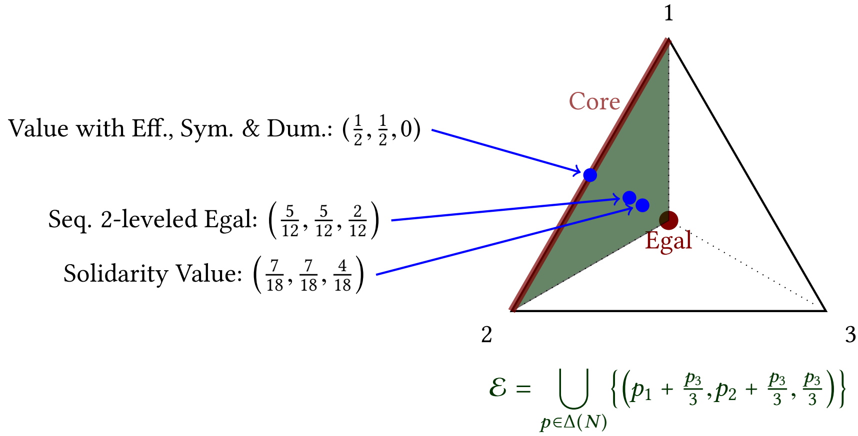


Fig. 4. $\mathcal{E}(N, v)$ with $V = \{1, 2\}$

The set of equilibrium outcomes is again the convex hull of the core and the egalitarian point. The payoff of the dummy player might be positive but less than that of any veto player. The set of equilibrium outcomes includes all the cooperative solutions with the axioms of *efficiency* and *dummy player property* and also includes some other alternative solutions without dummy player property, such as *sequentially 2-leveled egalitarianism* and *solidarity value*.

5 CONCLUDING REMARKS

In this paper, we analyzed strategic alliance behaviors and gradual agreement phenomena, introducing a new noncooperative bargaining model with strategic alliances. A general inefficiency result is provided: inefficient coalitions form as an intermediate bargaining step when players essential for an efficient coalition are involved. We characterized a necessary condition for delay in bargaining and uncovered the role of an essential player or a veto player in inefficiency.

In addition to efficiency, the effect of strategic alliances on *equality* could also be an important issue. For a non-unanimity simple game with a veto player, as Section 4 discussed, allowing strategic alliances significantly decreases inequality. For a general characteristic function form game, however, allowing intermediate coalition formation may cause inequality to increase. For instance, consider a three-player game with $v(N) = 1$, $v(\{1, 2\}) = \frac{1}{2}$, and $v(S) = 0$ otherwise. When players have no buyout option, a grand coalition always immediately forms and they split the unit surplus according to their recognition probabilities; however when they have buyout options, the two major players forms an intermediate coalition $\{1, 2\}$ with a positive probability, so that they take more payoffs than their recognition probabilities. The bargaining model with buyout options may provide a benchmark tool to understand how the *freedom of association* actually affects social inequality.

Although inefficiency generically occurs for high discount factors, it is worth noting that any equilibrium must be *asymptotically efficient*. As long as the current state is inefficient, the remaining players continue to bargain over the remaining surplus and hence any inefficient state transitions into an efficient state in a finite period. Thus, as the discount factor increases, two different effects on efficiency are intertwined: a strategic delay occurs more and more frequently, but it becomes less and less costly and hence inefficiency eventually disappears as a discount factor converges to 1.

Despite of asymptotic efficiency, measuring possible efficiency loss in equilibrium is important but it remains an open question. Computing the *price of anarchy* in coalition bargaining and analyzing its upper bound may provide an implication on the social cost of allowing strategic alliances or the *freedom of association*.

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A PROOFS

A.1 Proposition 3.1

First, in the following lemma, we characterizes the payoff vector in grand coalition equilibria. Though we assume zero-normalization on initial characteristic functions, but in non-initial states, the induced characteristic functions may not be zero-normalized. The following lemma does not rely on zero-normalization.

Lemma A.1. *Suppose* $(\mathbf{x}, \mathbf{q})$ *is a grand coalition equilibrium of* (N, v, p, δ). *For each* $i \in N$,

i) $u_i(\mathbf{x}, \mathbf{q}) = v_i + p_i(\bar{v} - v_N)$; *and*
ii) $x_i = v_i + \delta p_i(\bar{v} - v_N)$.

Proof. Since $(\mathbf{x}, \mathbf{q})$ is efficient, it must be $u_N(\mathbf{x}, \mathbf{q}) = \bar{v}$ and $x_N = \delta\bar{v}$. Take $i \in N$. i always forms a grand coalition and i is always included by other players' proposal as well. Thus, i's expected equilibrium payoff is

$$u_i(\mathbf{x}, \mathbf{q}) = p_i(\bar{v} - x_N) + x_i = p_i(1 - \delta)\bar{v} + x_i. \tag{4}$$

Consistency condition in Proposition 2.2 requires

$$x_i = (1 - \delta)v_i + \delta u_i(\mathbf{x}, \mathbf{q}) \tag{5}$$

and hence (4) yields the first result. Plugging the first result into (5), the second result follows. □

Proof of Proposition 3.1: For any two-player game, the statement is clearly true. As an induction hypothesis, suppose the statement is true for any less-than-n-player game. Now consider an n-player game (N, v, p, δ).

Step 1: $(\delta\bar{v}\mathbf{p}, \bar{\mathbf{q}})$ is an equilibrium.
For any non-initial state π, the strategy profile $(\delta\bar{v}\mathbf{p}, \bar{\mathbf{q}})$ constitutes an equilibrium in the subgame starting with π, due to the induction hypothesis and Lemma A.1. Now we verify $(\delta\bar{v}p, \bar{q})$ constitutes an equilibrium in the initial state. First, consistency condition is hold by Lemma A.1. Second, for each $i \in N$ and $S \subsetneq N_i$,

$$e_i(S, \delta\bar{v}\mathbf{p}) = x_i^{(i,S)} - x_S = \delta\bar{v}p^{(i,S)} - \delta\bar{v}p_S = 0.$$

However, $e_i(N, \delta\bar{v}\mathbf{p}) = \bar{v} - \delta\bar{v}p_N = (1 - \delta)\bar{v} > 0$, and hence forming a grand coalition satisfies optimality condition.

Step 2: An equilibrium payoff vector is always $\bar{v}p$.
Let $(\mathbf{x}, \mathbf{q})$ be an equilibrium. If $q = \bar{q}$, then Lemma A.1 implies $x = \delta\bar{v}p$, which yields the equilibrium payoff vector is $\bar{v}p$. Suppose that there exists $i \in N$ and $S \subsetneq N$ such that $q_i(S) > 0$. Since $(\mathbf{x}, \mathbf{q})$ is inefficient, it must be $x_N < \delta\bar{v}$. Player i's optimality condition requires that $x_i^{(i,S)} - x_S \geq \bar{v} - x_N$, and hence the induction hypothesis implies that $\delta\bar{v}p_S - x_S \geq \bar{v} - x_N$. Putting $x_N < \delta\bar{v}$, we have

$$x_S < \delta\bar{v}p_S - (1 - \delta)\bar{v}. \tag{6}$$

On the other hand, for each $j \in S$, let $Q_j = \sum_{k \in N} p_k \sum_{S \subseteq N} q_k(S) \mathbb{1}(j \in S)$. Player j's continuation payoff is no less than her payoff from making an offer with the grand coalition, and hence we have

$$\begin{aligned} u_j(\mathbf{x}, \mathbf{q}) &\geq p_j(\bar{v} - x_N) + Q_j x_j + (1 - Q_j)\delta p_j \bar{v} \\ &> p_j\bar{v} + Q_j(x_j - \delta p_j\bar{v}). \end{aligned} \tag{7}$$

Since $\delta u_j(\mathbf{x}, \mathbf{q}) = x_j$, rearranging (7), we have $x_j > \delta p_j\bar{v}$. Since this inequality holds for any $j \in S$, by summing this over S, it follows $x_S > \delta\bar{v}p_S$, which contradicts to (6). Therefore, for all $i \in N$, it must be $q_i(N) = 1$ in any equilibrium. However, this contracts to $q_i(S) > 0$ with $S \subsetneq N$. □

A.2 Proposition 3.2

PROOF OF PROPOSITION 3.2: Since (N, v) is not a unanimity game, there exists $S \subsetneq N$ such that $v(S) > 0$. By superadditivity, for all $k \in N \setminus S$, we have $v(N \setminus \{k\}) > 0$. Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Take any $i \in S$. Player i's optimality condition requires $e_i(N, \mathbf{x}) \geq e_i(N \setminus \{k\}, \mathbf{x})$, or equivalently,

$$x_i^{(i,N)} - x_N \geq x_i^{(i,N\setminus\{k\})} - x_N + x_k. \tag{8}$$

First, Lemma A.1 implies that $x_i^{(i,N)} = \bar{v}$ and $x_k = \delta p_k \bar{v}$. Secondly, i's $(N \setminus \{k\})$-formation induces a two-player game, and hence Lemma A.1 yields $x_i^{(i,N\setminus\{k\})} = v(N \setminus \{k\}) + \delta(1 - p_k)\left(\bar{v} - v(N \setminus \{k\})\right)$. Plugging x_k, $x_i^{(i,N)}$, and $x_i^{(i,N\setminus\{k\})}$ into (8), we have

$$\bar{v} \geq v(N \setminus \{k\}) + \delta(1 - p_k)\left(\bar{v} - v(N \setminus \{k\})\right) + \delta p_k \bar{v}.$$

Rearranging the terms, it follows

$$(1 - \delta)\bar{v} \geq (1 - \delta(1 - p_k))v(N \setminus \{k\}). \tag{9}$$

As $\delta \to 1$, the left-hand side of (9) converges to zero; while the right-hand side is strictly positive uniformly on δ, which yields a contradiction. More precisely, let

$$\bar{\delta} = \frac{v(N \setminus \{k\})}{(1 - p_k)v(N \setminus \{k\}) + p_k \bar{v}},$$

which is strictly less than 1. Then for all $\delta > \bar{\delta}$, (9) does not hold, and hence forming a grand coalition is not optimal for i. Therefore, $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium for any $\delta > \bar{\delta}$. □

A.3 Lemma 3.5

PROOF OF LEMMA 3.5: Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Let $\bar{x} = \min_{S \in \mathbf{E}} x_S$ and $\mathbf{E}^* = \operatorname{argmin}_{S \in \mathbf{E}} x_S$. For any $i \in N$, since $u_i \geq p_i e(N) + p_i x_i$, it follows $x_i \geq \frac{p_i}{1 - \delta p_i}(1 - \delta) > 0$ and hence $\mathbf{E}^* \subseteq \mathbf{E}^m$. Since $(\mathbf{x}, \mathbf{q})$ is efficient, for all $i \in N$ and $S \subseteq N$, $q_i(S) > 0$ implies $K \subseteq S$; and for all $k \in K$ and $S \subseteq N$, $q_k(S) > 0$ implies $S \in \mathbf{E}^*$. Therefore, for any $k \in K$, $u_k(\mathbf{x}, \mathbf{q}) = p_k(\bar{v} - \bar{x}) + x_k$. Summing this over K, we have $u_K(\mathbf{x}, \mathbf{q}) = p_K(\bar{v} - \bar{x}) + x_K$, or equivalently,

$$(1 - \delta)x_K = \delta p_K(\bar{v} - \bar{x}) = p_K(x_N - \delta \bar{x}) > p_K(x_N - \bar{x}). \tag{10}$$

Take any $S \in \mathbf{E}^*$ so that $x_S = \bar{x}$. First, suppose $K \cup (N \setminus S) \notin \mathbf{E}$. For any $A \subseteq N \setminus S$, monotonicity implies that $K \cup A \notin \mathbf{E}$. Since $S \in \mathbf{E}^m$, for any $j \in S$, it follows that $S \setminus \{j\} \notin \mathbf{E}$. Therefore, $S = K$ and hence $\bar{x} = x_K$ as desired. Next, suppose $K \cup (N \setminus S) \in \mathbf{E}$. Since $S \in E^*$, $x_{K \cup (N \setminus S)} \geq \bar{x}$. Note that $K \subseteq S$ and hence $K \cap (N \setminus S) = \emptyset$. Thus it follows that $x_K + x_{N \setminus S} \geq \bar{x}$, or equivalently, $x_N - \bar{x} \geq \bar{x} - x_K$. Furthermore, $K \subseteq S$ implies $\bar{x} - x_K \geq 0$. Thus (10) yields

$$(1 - \delta)x_K = \delta p_K(\bar{v} - \bar{x}) > p_K(\bar{x} - x_K) \geq 0. \tag{11}$$

Thus, as $\delta \to 1$, $x_K \to \bar{x}$ and $\bar{x} \to \bar{v}$, as desired. □

A.4 Theorem 3.4

A.4.1 Case 1: $K \notin \mathbf{E}$.

If $K \notin \mathbf{E}$, then the set of non-essential players can be a collective essential player in the subsequent period by forming a coalition, after which there is a unique efficient coalition. Lemma A.2 shows that if there is a unique efficient coalition, then each player gets at least some positive portion of her marginal contribution to the grand coalition in addition to her stand-alone value.

Lemma A.2. *Suppose $v(S) < \bar{v}$ for all $S \subsetneq N$. If $(\mathbf{x},\mathbf{q})$ is an equilibrium of (N,v,p,δ), for any $i \in N$*

$$u_i(\mathbf{x},\mathbf{q}) \geq v_i + \delta^{|N|-2} p_i \left(\bar{v} - v(N \setminus \{i\})\right).$$

Proof. If $|N| = 2$, then (N,v) is a unanimity game and Proposition 3.1 proves this case. As induction hypothesis, suppose the result holds for any less-than-n-player game. Consider (N,v) with $|N| = n$ and $\forall S \subsetneq N$ $v(S) < \bar{v}$. Take any $i \in N$.

$$\begin{aligned} u_i(\mathbf{x},\mathbf{q}) &\geq p_i \cdot 0 + \sum_{j\in N} p_j \sum_{S\subseteq N} q_j(S) \left(\mathbb{1}(i \in S)x_i + \mathbb{1}(i \notin S)x_i^{(j,S)}\right) \\ &\geq Q_i\left((1-\delta)v_i + \delta u_i(\mathbf{x},\mathbf{q})\right) \\ &\quad +(1-Q_i)\left((1-\delta)v_i + \delta\left(v_i + \delta^{n-3}p_i^{\pi}\left(\bar{v} - v(N\setminus\{i\})\right)\right)\right) \\ &= \delta Q_i u_i(\mathbf{x},\mathbf{q}) + (1-\delta Q_i)v_i + (1-Q_i)\delta^{n-2}(\bar{v} - v(N\setminus\{i\})), \end{aligned}$$

where $Q_i = \sum_{j\in N} p_j \sum_{S\subseteq N} q_j(S)\mathbb{1}(i \in S)$ and the second inequality comes from the induction hypothesis. Rearranging the terms, the inequality yields

$$\begin{aligned} u_i(\mathbf{x},\mathbf{q}) &\geq v_i + \frac{1-Q_i}{1-\delta Q_i}\delta^{n-2}(\bar{v} - v(N\setminus\{i\})) \\ &\geq v_i + \delta^{n-2}(\bar{v} - v(N\setminus\{i\})), \end{aligned}$$

as desired. □

Based on Lemma A.2, under assuming an efficient equilibrium, we show that a non-essential player can be better off by forming a coalition which consists of non-essential players, rather than immediately forming an efficient coalition. However, this is a contradiction and proves the impossibility of an efficient equilibrium.

Proof of Theorem 3.4 (Case 1: $K \notin \mathbf{E}$)
If $K \notin \mathbf{E}$, then $v(K) < \bar{v}$. Suppose that $(\mathbf{x},\mathbf{q})$ is an efficient equilibrium. Take any $i \in N \setminus K$. Since $(\mathbf{x},\mathbf{q})$ is efficient, there exists $A \neq \emptyset$ such that $K \cup A \in \mathbf{E}$ and $q_i(K \cup A) > 0$. Player i's optimality condition requires that $e_i(K \cup A, \mathbf{x}) \geq e_i(N \setminus K, \mathbf{x})$, which is

$$\bar{v} - x_K - x_A \geq x_i^{(i,N\setminus K)} - x_N + x_K.$$

After i's $(N \setminus K)$-formation, since there is only one efficient coalition, Lemma A.2 yields that

$$\bar{v} - x_K - x_A \geq v(N\setminus K) + \delta^{|K|-2}(1-p_K)(\bar{v} - v(K)) - x_N + x_K.$$

Since $x_N = \delta\bar{v}$ and $v(N \setminus K) \geq 0$, rearranging the terms, we have

$$(1+\delta)\bar{v} - 2x_K - x_A \geq \delta^{|K|-2}(1-p_K)(\bar{v} - v(K)). \tag{12}$$

Due to Lemma 3.5, as $\delta \to 1$, the left-hand side of (12) converges to 0, while the right-hand side converges to $(1-p_K)(\bar{v} - v(K)) > 0$, which yields a contradiction. □

A.4.2 Case 2: $K \in \mathbf{E}$ and $\exists k' \in K$ $v(N \setminus \{k'\}) > 0$.

We will show that at least one of essential players can be better off by excluding the other essential player if all the players are supposed to play efficient strategies.

Lemma A.3. *If $K \in \mathbf{E}$, then $\mathbf{E}^m = \{K\}$, $K \cup D = N$, $K \cap D = \emptyset$, and $|K| \geq 2$. In addition, if $D = \emptyset$, then $K = N$ and $\mathbf{E}^m = \mathbf{E}$.*

Proof. Suppose $K \in \mathbf{E}$. Take any $S \in \mathbf{E}$. Since $K = \cap_{S' \in \mathbf{E}} S'$, we have $S \cap K = K$, and hence $K \subseteq S$, which implies $\mathbf{E}^m = \{K\}$. It follows that $D \equiv N \setminus (\cup \mathbf{E}^m) = N \setminus K$, and hence $K \cup D = N$ and $K \cap D = \emptyset$. If $|K| = 0$, then $K = \emptyset \in \mathbf{E}$. By monotonicity, for any $i \in N$, $\{i\} \in \mathbf{E}$, which violates zero-normalization. If $|K| = 1$, say $K = \{k\}$, then $\{k\} \in \mathbf{E}$, which violates zero-normalization. Therefore, we conclude $|K| \geq 2$. The second part is trivial. □

Lemma A.4 provides a lower bound of a cutoff value for each essential player under assuming an efficient equilibrium.

Lemma A.4. *Suppose* $(\mathbf{x}, \mathbf{q})$ *is an equilibrium of* (N, v, p, δ). *If* $K \in \mathbf{E}$, *then, for any* $k \in K$,

$$x_k = \frac{\delta}{1-\delta} p_k (\bar{v} - x_K) = \frac{p_k}{p_K} x_K \geq \delta p_k \bar{v}.$$

Proof. Since $(\mathbf{x}, \mathbf{q})$ is efficient, for each $i \in N$, $q_i(S) > 0$ implies $K \subset S$. Thus, for any $k \in K$, we have $u_k(\mathbf{x}, \mathbf{q}) = p_k(\bar{v} - x_K) + x_k$, and hence

$$(1-\delta) x_k = \delta p_k (\bar{v} - x_K), \tag{13}$$

which implies the first equality. Summing (13) over K, we have $(1-\delta)x_K = \delta p_K(\bar{v} - x_K)$. Plugging this into (13), we have the second equality. Since $u_N(\mathbf{x}, \mathbf{q}) \leq \bar{v}$, we have $x_K \leq x_N \leq \delta \bar{v}$ and hence $(1-\delta)x_k \geq \delta p_k (1-\delta)\bar{v}$, which implies the inequality part. □

Proof of Theorem 3.4 (Case 2: $K \in \mathbf{E}$ and $\exists k' \in K\ v(N \setminus \{k'\}) > 0$)
By Lemma A.3, $|K| \geq 2$ and, hence, we can take $k \in K$ such that $k \neq k'$. Let $(\mathbf{x}, \mathbf{q})$ be an efficient equilibrium. Player k's optimality condition implies that $e_k(K, \mathbf{x}) \geq e_k(N \setminus \{k'\}, \mathbf{x})$, that is,

$$\bar{v} - x_K \geq x_k^{(k, N\setminus\{k'\})} - x_{N\setminus\{k'\}}. \tag{14}$$

Since k's $(N \setminus \{k'\})$-formation yields a two-player game, k's value in the subsequent state is

$$x_k^{(k, N\setminus\{k'\})} = v(N \setminus \{k'\}) + \delta(1 - p_{k'})(\bar{v} - v(N \setminus \{k'\})). \tag{15}$$

Plugging (15) into (14), we have

$$\begin{aligned}
(1 - \delta(1-p_{k'}))\bar{v} - x_K &\geq (1 - \delta(1-p_{k'}))v(N \setminus \{k'\}) - x_N + x_{k'} \\
(1 + \delta p_{k'})\bar{v} - x_K &\geq (1 - \delta(1-p_{k'}))v(N \setminus \{k'\}) + x_{k'} \\
\bar{v} - x_K &\geq (1 - \delta(1-p_{k'}))v(N \setminus \{k'\}),
\end{aligned} \tag{16}$$

where the second line comes from efficiency $x_N = \delta \bar{v}$; and the third line is due to Lemma A.4. However, by Lemma 3.5, the left-hand side of (16) converges to 0 as $\delta \to 1$; while the right-hand side converges to $p_{k'} v(N \setminus \{k'\}) > 0$, which yields a contradiction. □

A.4.3 Case 3: $K \in \mathbf{E}$ *and* $\forall k' \in K\ v(N \setminus \{k'\}) = 0$.

If $v(N \setminus \{k'\}) = 0$ for all $k' \in K$, then it may not be profitable for an essential player k to exclude the other essential player k' as in Case 2. We show that it is profitable for $k \in K$ to form a coalition with non-essential players rather than forming an efficient coalition.

Proof of Theorem 3.4 (Case 3: $K \in \mathbf{E}$ and $\forall k' \in K\ v(N \setminus \{k'\}) = 0$)
If $D = \emptyset$, then Lemma A.3 implies that there exists a unique efficient coalition, and hence Proposition 3.2 completes the proof. Now we assume that $D \neq \emptyset$. Again Lemma A.3 yields $K \cup D = N$ and

$p_K < 1$. Take any $k \in K$. Due to Lemma A.3, $|K| \geq 2$ and hence $p_K > p_k$. Thus $(1-p_K)(p_K-p_k) > 0$. Rearranging terms and using $p_K + p_D = 1$, it follows that

$$1 - p_D - p_k < 1 - \frac{p_k}{p_K}. \tag{17}$$

Now suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Player k's optimality condition implies that $e_k(K, \mathbf{x}) \geq e_k(D \cup \{k\}, \mathbf{x})$, that is,

$$\bar{v} - x_K \geq x_k^{(k, D\cup\{k\})} - x_D - x_k. \tag{18}$$

After k's $(D \cup \{k\})$-formation, there is only one efficient coalition, and hence, due to Lemma A.2, (18) yields

$$\bar{v} - x_K \geq v(D \cup \{k\}) + \delta^{|K|-1}(p_D + p_k)(\bar{v} - v(K \setminus \{k\}) - x_D - x_k. \tag{19}$$

Since $v(K \setminus \{k\}) = 0$ and $x_k = \frac{p_k}{p_K} x_K$ due to Lemma A.4, (19) yields

$$\left(1 - \delta^{|K|-1}(p_D + p_k)\right) \bar{v} + x_D \geq \left(1 - \frac{p_k}{p_K}\right) x_K. \tag{20}$$

Due to Lemma 3.5, as $\delta \to 1$, (20) requires that $1 - p_D - p_k \geq 1 - \frac{p_k}{p_K}$, which contradicts to (17). □

A.5 Proposition 4.3

Before proving the proposition, we show that the two players whose recognition probabilities are not the greatest must get the same payoff for sufficiently high discount factors in the following lemma.

Lemma A.5. *Let (x, q) be an equilibrium of a three-player simple game with no veto player. For any p with $p_1 \geq p_2 \geq p_3$, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, $x_1 \geq x_2 = x_3$.*

Proof. Note that any equilibrium (x, q) is efficient so that $x_N = \delta$. It is also clear that $x_1 \geq \frac{\delta}{3}$ and hence

$$x_1 = \delta - x_2 - x_3 \geq \frac{\delta}{3}. \tag{21}$$

Suppose by way of contraction that $x_2 > x_3$. Since $\sum_{S:3\in S} \sum_{i\in N} p_i q_i(S) = 1$, we have $x_3 = \delta(p_3(1 - x_2 - x_3) + x_3)$ or

$$x_3 = \frac{\delta p_3(1 - x_2)}{1 - \delta(1 - p_3)}. \tag{22}$$

By (21) and (22), we have

$$\delta - x_2 - \frac{\delta p_3(1 - x_2)}{1 - \delta(1 - p_3)} > \frac{\delta}{3}. \tag{23}$$

Since the left-hand side of (23) converges to 0 as $\delta \to 1$, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, (23) yields a contraction. □

Proof of Proposition 4.3. Due to Lemma A.5, there exists $\bar{\delta}$ such that for all $\delta > \bar{\delta}$, $x_1 \geq x_2 = x_3$, which implies that $e(\{2,3\}, x) \geq e(\{1,2\}, x) = e(\{1,3\}, x)$.

Case 1: $e(\{2,3\}, x) > e(\{1,2\}, x) = e(\{1,3\}, x)$.

Since $\sum_{S:1\in S} \sum_{i\in N} p_i q_i(S) = p_1$, we have

$$x_1 = \delta(p_1(1 - x_1 - x_2) + p_1 x_1) = \delta p_1(1 - x_2). \tag{24}$$

Since any equilibrium is efficient, $x_N = \delta$ and hence $x_2 = x_3 = \frac{\delta - x_1}{2}$. Thus, (24) implies

$$x_1 = \frac{\delta(2-\delta)p_1}{2-\delta p_1} \quad \text{and} \quad x_2 = x_3 = \frac{\delta(1-p_1)}{2-\delta p_1}. \tag{25}$$

The condition $e(\{2,3\},x) > e(\{1,2\},x)$ requires $x_1 > x_2$, or $\delta < 3 - \frac{1}{p_1}$ due to (25). If $p_1 \geq \frac{1}{2}$, then $3 - \frac{1}{p_1} \leq 1$ and (25) consists of the equilibrium payoffs for sufficiently high δ, which completes the first part. If $p_1 < \frac{1}{2}$, then it contradicts to the condition $e(\{2,3\},x) > e(\{1,2\},x)$ for sufficiently high δ.

Case 2: $e(\{2,3\},x) = e(\{1,2\},x) = e(\{1,3\},x)$.

It must be $x_1 = x_2 = x_3 = \frac{\delta}{3}$ for sufficiently high δ. Thus, we have

$$x_1 = \delta(p_1(1-x_1-x_2) + (p_1 + p_2q_2(\{1,2\}) + p_3q_3(\{1,3\}))x_1),$$

which implies

$$x_1 = \frac{\delta p_1(3-2\delta)}{3(1-\delta(p_1 + p_2q_2(\{1,2\}) + p_3q_3(\{1,3\})))}.$$

Then, $x_1 = \frac{\delta}{3}$ yields $p_1(3-2\delta) = 1 - \delta(p_1 + p_2q_2(\{1,2\}) + p_3q_3(\{1,3\}))$, or

$$p_1 = \frac{1 - \delta(p_2q_2(\{1,2\}) + p_3q_3(\{1,3\}))}{3-\delta}. \tag{26}$$

If $p_1 < \frac{1}{2}$, then it must be $q_2(\{1,2\}) + q_3(\{1,3\}) > 0$ as $\delta \to 1$. If $p_1 > \frac{1}{2}$, (26) implies $p_2q_2(\{1,2\}) + p_3q_3(\{1,3\}) < 0$ for sufficiently high δ, which yields a contradiction. □

A.6 Proposition 4.4

In the following lemma, we show that the two non-veto players get the same payoff in equilibrium and the veto player gets at least as much as the others.

Lemma A.6. *Let (x,q) be an equilibrium of a three-player simple game with $V = \{1\}$. For any p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, $x_1 \geq x_2 = x_3$.*

Proof. Suppose $x_2 > x_3$. We show a contradiction in each possible case.

Case 1: $e(\{23\},x) \geq e(\{13\},x) > e(\{12\},x)$.

The expected payoffs of player 2 and player 3 are:

$$\begin{aligned} x_2 &= \delta(p_2(\delta(p_2+p_3) - x_2 - x_3) + (p_2+p_3)x_2); \\ x_3 &= \delta(p_3(\delta(p_2+p_3) - x_2 - x_3) + x_3), \end{aligned}$$

which implies that

$$x_2 = \frac{\delta^2(1-\delta)p_2(p_2+p_3)}{1-\delta+p_1p_3\delta^2} \quad \text{and} \quad x_3 = \frac{\delta^2 p_3(p_2+p_3)(1-\delta(p_2+p_3))}{1-\delta+p_1p_3\delta^2}.$$

As $\delta \to 1$, x_2 converges to 0 and x_3 converges to p_2+p_3. Thus, for sufficiently high δ, it contradicts to $x_2 > x_3$.

Case 2: $e(\{13\},x) \geq e(\{23\},x) > e(\{12\},x)$.

The expected payoffs of player 1 and player 3 are:

$$\begin{aligned} x_1 &= \delta(p_1(1-x_1-x_3) + (p_1+p_3)x_1); \\ x_3 &= \delta(p_3(1-x_1-x_3) + x_3), \end{aligned}$$

which implies that

$$x_3 = \frac{\delta p_3(1-\delta(p_1+p_3))}{1-\delta+p_2p_3\delta^2}.$$

As $\delta \to 1$, x_3 converges to 1, which contradicts to $x_2 > x_3$ for sufficiently high δ.

Case 3: $e(\{13\},x) \geq e(\{12\},x) > e(\{23\},x)$.

It must be $q_1(\{1,3\}) = q_3(\{1,3\}) = q_2(\{1,2\}) = 1$, which implies that a winning coalition must form immediately, which contradicts to Theorem 3.4. □

Proof of Proposition 4.4.

i) Suppose $q_{23} = q_{32} = 1$. It must be $e(\{2,3\},x) \geq e(\{1,2\},x)$, or $x_1 - x_2 \geq p_1$. Since $x_2 = x_3$ by Lemma A.6, the veto player's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2 x_1^{(2,\{2,3\})} + p_3 x_1^{(3,\{2,3\})} \\ &= p_1\left(1 - \left(\frac{1}{2} - \frac{x_1}{2}\right)\right) + (1-p_1)p_1, \end{aligned}$$

which yields (2). The condition $x_1 - x_2 \geq p_1$ requires that $\frac{p_1(3-2p_1)}{2-p_1} - \frac{1-p_1}{3} \geq p_1$. Solving this inequality, p_1 must satisfy $-2p_1^2 + 3p_1 - 1 \geq 0$, or $\frac{1}{2} \leq p_1 \leq 1$. This completes the proof of the first part.

ii) Suppose $0 < q_{23} < 1$ and $0 < q_{32} < 1$. It must be $e(\{2,3\},x) = e(\{1,2\},x) = e(\{1,3\},x)$, or $x_1 - x_2 = p_1 = x_1 - x_3$. Solving these equations with $x_N = 1$, we have (3). In this case, the veto player's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2\left(q_{21}x_1 + q_{23}x_1^{(2,\{2,3\})}\right) + p_3\left(q_{31}x_1 + q_{32}x_1^{(3,\{2,3\})}\right) \\ &= p_1(1-x_2) + rx_1 + (1-r)p_1 - p_1^2, \end{aligned} \tag{27}$$

where $r = p_2q_{21} + p_3q_{31} > 0$ is the probability that the veto player is included in the proposed coalition. Plugging (3) into (27), it follows that

$$\frac{1+2p_1}{3} = p_1\left(1 - \frac{1-p_1}{3}\right) + r\frac{1+2p_1}{3} + (1-r)p_1 - p_1^2,$$

which yields $r = 1 - 2p_1$. Since $r > 0$, it must be $r = 1 - 2p_1 > 0$, or $p_1 < \frac{1}{2}$. This completes the proof of the second part. □

A.7 Proposition 4.5

Following Lemma shows that the excess surpluses of two-player coalitions are all the same and strictly greater than that of other coalitions in equilibrium.

Lemma A.7. *Let (x,q) be an equilibrium of a three-player simple game with $V = \{1,2\}$. For any p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$,*

$$\operatorname*{argmax}_{S \subseteq N} e(S,x) = \{\{1,2\},\{1,3\},\{2,3\}\}.$$

Proof.

Step 1: We show $q_i(N) = 0$ for all $i \in N$.

Since $x_3 > 0$ for all $\delta \in (0,1)$, it must be $e(\{1,2\},x) > e(N,x)$ and hence $q_1(\{1,2\}) = q_2(\{1,2\}) = 1$. If $q_3(N) = 1$, then the equilibrium is efficient which contradicts to Theorem 3.4. Thus it must be $q_3(N) < 1$, which implies either $e(\{1,3\},x) \geq e(N,x)$ or $e(\{2,3\},x) \geq e(N,x)$. Without loss of generality, suppose $e(\{1,3\},x) \geq e(N,x)$. If the inequality is strict, then $q_3(N) = 0$ and the proof is completed. Now suppose it holds with equality. If $e(\{2,3\},x) > e(\{1,3\},x) = e(N,x)$, then again we have $q_3(N) = 0$, which completes the proof. Thus it must be $e(\{1,3\},x) = e(N,x) \geq e(\{2,3\},x)$, which implies $\delta(p_1+p_3) - x_1 - x_3 = 1 - x_N$ and $1 - x_N \geq \delta(p_2+p_3) - x_2 - x_3$, or equivalently, $x_2 = \delta p_2 + (1-\delta)$ and $x_1 \leq \delta p_1 + (1-\delta)$.

Note that $x_3 \le \delta(p_3(1-x_N)+p_3x_3)$ and hence $x_3 = \frac{\delta p_3}{1-\delta p_3}(1-x_N)$. Putting altogether, we have $x_N \le \delta p_1 + (1-\delta) + \delta p_2 + (1-\delta) + \frac{\delta p_3}{1-\delta p_3}(1-x_N)$, or

$$x_N \le (1-\delta p_3)\left[\delta(1-p_3)+2(1-\delta)\right]+\delta p_3. \tag{28}$$

Note the right-hand side of (28) converges to $1-p_3(1-p_3)<1$ as $\delta\to 1$. However, (28) is a contradiction for sufficiently high δ, because $x_N \to 1$ as $\delta \to 1$.

Step 2: We show that $|\operatorname{argmax}_S e(S,x)| \ge 2$.

Suppose by way of a contradiction, $|\operatorname{argmax}_S e(S,x)| = 1$. There must be $\{i,j\}$ such that

$$e(\{i,j\},x) > e(\{i,k\},x) \quad \text{and} \quad e(\{i,j\},x) > e(\{j,k\},x). \tag{29}$$

Since $1-x_S \ge e(S,x) \ge \delta p_S - x_S$ for any $S\subset N$, (29) implies

$$x_i < 1-\delta(1-p_i)+x_k, \text{ and} \tag{30}$$

$$x_j < 1-\delta(1-p_j)+x_k. \tag{31}$$

On the other hand, since $\sum_{S:k\in S}(q_i(S)+q_j(S))=0$, (29) implies

$$x_k < \delta(p_k e(\{i,j\},x)+p_kx_k) \le \delta p_j(1-x_N+2x_k) \le \delta p_j(1-\delta^2+2x_k)$$

and hence

$$x_k < \frac{\delta p_k}{1-2\delta p_k}(1-\delta^2). \tag{32}$$

From (30), (31), and (32), we have

$$x_N < p_i+p_j+2(1-\delta)+3x_k < p_i+p_j+2(1-\delta)+3\frac{\delta p_k}{1-2\delta p_k}(1-\delta^2). \tag{33}$$

Since the right-hand side of (33) converges to $p_i+p_j<1$ as $\delta\to 1$, it is a contradiction again for sufficiently high δ.

Step 3: We show that $\operatorname{argmax}_S e(S,x) \neq \{\{1,3\},\{2,3\}\}$.

Suppose $\operatorname{argmax}_S e(S,x) = \{\{1,3\},\{2,3\}\}$, Then $x_N=\delta^2$ and we have

$$\begin{aligned} x_1 &= \delta(p_1e(\{1,3\},x)+(p_1+p_3q_3(\{1,3\}))x_1) \\ &= \delta(p_1\underbrace{\left[\delta(p_2+p_3)-x_N+x_1\right]}_{e(\{1,3\},x)=e(\{2,3\},x)}+(p_1+p_3q_3(\{1,3\}))x_1) \\ &= \delta(p_1\left[\delta(1-p_1)-\delta^2\right]+(2p_1+p_3q_3(\{1,3\}))x_1), \end{aligned}$$

and hence

$$\left[1-\delta(2p_1+p_3q_3(\{1,3\}))\right]x_1 = \delta^2p_1(1-\delta-p_1). \tag{34}$$

For sufficiently high δ, the right-hand side of (34) is strictly negative and hence it must be

$$\delta(2p_1+p_3q_3(\{1,3\}))>1. \tag{35}$$

Similarly, by solving x_2, we have

$$\delta(2p_2+p_3q_3(\{2,3\}))>1. \tag{36}$$

Recall that $q_3(\{1,3\})+q_3(\{2,3\})=1$ due to Step 1. By (35) and (36), we have $\delta(p_1+p_2)>1$, which yields a contradiction.

Step 4: We show that $\operatorname{argmax}_S e(S,x) = \{\{1,2\},\{1,3\},\{2,3\}\}$.

A cutoff strategy equilibrium must exists, it suffices to show $\operatorname{argmax}_S e(S,x) \neq \{\{1,2\},\{1,3\}\}$ and $\operatorname{argmax}_S e(S,x) \neq \{\{1,2\},\{2,3\}\}$. Without loss of generality, suppose $\operatorname{argmax}_S e(S,x) = \{\{1,2\},\{1,3\}\}$. Then

$$x_N = (p_1 q_1(\{1,2\}) + p_2)\delta + (p_1 q_1(\{1,3\}) + p_3)\delta^2 \tag{37}$$

and we have

$$\begin{aligned} x_2 &= \delta(p_2 e(\{1,2\},x) + (p_2 + p_1 q_1(\{1,2\}))x_2) \\ &= \delta(p_2\,[\delta(p_1 + p_3) - x_N + x_2] + (p_2 + p_1 q_1(\{1,2\}))x_2) \\ &= \delta(p_2\,[\delta(1 - p_2) - x_N] + (2p_2 + p_1 q_1(\{1,2\}))x_2). \end{aligned}$$

Rearranging the terms, we have

$$[1 - \delta(2p_2 + p_1 q_1(\{1,2\}))]\, x_2 = \delta p_2(\delta - x_N - \delta p_2). \tag{38}$$

From (37), note that $x_N \to 1$ as $\delta \to 1$, and hence the right-hand side of (38) is strictly negative for sufficiently high δ. Thus, we have

$$\delta(2p_2 + p_1 q_1(\{1,2\})) > 1. \tag{39}$$

On the other hand, we have

$$\begin{aligned} x_3 &= \delta(p_3 e(\{1,3\},x) + (p_3 + p_1 q_1(\{1,3\}))x_3) \\ &= \delta(p_3(1 - x_N) + (2p_2 + p_1 q_1(\{1,3\}))x_3) \end{aligned}$$

and hence $[1 - \delta(2p_3 + p_1 q_1(\{1,3\}))]\, x_3 = \delta p_3(1 - x_N)$. Since the right-hand side converges to 0 as $\delta \to 1$, it must be either $2p_3 + p_1 q_1(\{1,3\}) \to 1$ or $x_3 \to 0$. If $2p_3 + p_1 q_1(\{1,3\}) \to 1$ then it yields a contradiction to (39) for sufficiently high δ. Now assume $x_3 \to 0$. Note that $e(\{1,2\},x) = e(\{1,3\},x) > e(\{2,3\},x)$ implies $x_2 = 1 - \delta(1-p_2) + x_3$ and $x_1 < 1 - \delta(1-p_1) + x_3$. Thus, we have $x_N = 2(1-\delta) + (p_1 + p_2) + 3x_3$, which converges to $p_1 + p_2 < 1$, and hence it contradicts to (37) for sufficiently high δ. □

Proof of Proposition 4.5. By Lemma A.7, for sufficiently high δ, we have $e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x)$, which implies

$$x_1 - \delta p_1 = x_2 - \delta p_2 = x_3 + (1 - \delta), \tag{40}$$

and hence $x_N = 2(1-\delta) + \delta(1 - p_3) + 3x_3$. Since $\delta^2 \leq x_N \leq \delta$, we have $\lim_{\delta \to 1} x_N = 1$, which implies x_3 converges to $\frac{p_3}{3}$. Plugging x_3 into (40), we have the desired result. □