One-step-ahead Implementation

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Abstract

The paper examines problems of implementing social choice objectives in a dynamic environment, in which a society can achieve its objective in a one-step-ahead manner. The social objective that a society wants to achieve is summarized in a social choice function (SCF) that maps each state of the world into a *dynamic process* mapping every history into a socially desirable outcome. This social process is *one-step-ahead implementable* if there exists a process of one-period game forms (one game form after each outcome history), each of which generates a social outcome only at one given period after a given history, such that at each state of the world there is a subgame-perfect Nash equilibrium in which the social objective is fulfilled *at every period*, *after every history*, as a unique equilibrium outcome process. The paper identifies necessary conditions for SCFs to be one-step-ahead implemented, the *folding condition* and *one-step-ahead Maskin monotonicity*, and shows that they are also sufficient under auxiliary conditions when there are three or more agents. Finally, it provides an account of welfare implications of one-step-ahead implementability in the contexts of trading decisions and voting problems.

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1. Introduction

The theory of implementation investigates the goals that a planner/society can achieve when these goals depend on private information held by various agents. The problem of the planner is to design a mechanism or game form in which the agents' incentives dovetail to an equilibrium outcome that coincides with the planner's goal. When such a mechanism exists, his goal is fully implementable. This paper studies full implementation problems in a dynamic environment, in which:

- A finite number of agents interacts for a finite number of periods/stages and they commonly observe the state (of the world) before starting interacting.[1]
- A period-t environment consists of a set of agents, a set of period-t outcomes and agents' preferences over those outcomes. An agent's preferences over period-t outcomes endogenously depend on planners' decisions taken in the past as well as on planners' future goals.
- A period-t planner aims to solve his implementation problem by devising a one-shot mechanism (one after each outcome history) which asks agents to report only the information pertaining to his problem and which gives the agents the appropriate incentives so that a period-t "socially desirable" outcome results from the strategic behavior of the agents.
- The period-t planner concerns himself only about his implementation problem, aims to implement a socially desirable period-t outcome after any outcome history - *even* after off-equilibrium histories - and cannot punish agents over periods.[2]

Many real-world allocation problems have the above dynamic structure. Think of a group of parents with school-age children who move to a new city. There seems to be an obvious order of priorities for each family: first, to secure a property and then a school-seat. The families' rankings of schools are established endogenously by the outcome of the housing market, whereas the families' willingness to pay for a property depends on the allocation rules set by the school authority. The school authority aims to make a socially desirable matching of pupils to schools based on the rakings submitted by the families as well as on where the families reside. Moreover, the role of the housing market maker may be to facilitate the trade of houses but certainly not to make a commitment related to the outcome of the school admissions problem.

One can also observe the above structure in situations in which one authority handles problems over time but its identity changes over time, or its goal is achieved only in a one-step-ahead manner. For example, in democratic societies, the identity of governments changes over time due to periodic elections and the current government can decide and

[1] We assume finite periods for the sake of simplicity. Our framework extends to infinite periods by imposing a Markovian kind of refinement condition: Strategies depend only on outcome histories.

[2] Note that we do not require the existence of a period-t planner who is someone other than the agents. Indeed, there may not be such entity, as in the case of the contract relationship. In situations like these, contracting parties designs a period-t contract (period-t mechanism) with the goal to incentivizing themselves in pursuit of a commonly agreed goal.

execute only current policy variables at hand (see, e.g., Persson and Tabellini, 2000; Krusell et al., 1997). Also, in a market, the role of the market maker is to facilitate trade period-by-period but not to make a commitment related to future trading activities, or to enforce them over time (see, e.g., Radner, 1972, 1982; and Prescott and Mehra; 1980).

We thus study the following type of implementation problems. (i) A social choice objective is summarized in a social choice function (SCF) that maps each state into a dynamic process, which maps every outcome history into a socially desirable outcome at each given period.[3] (ii) A dynamic mechanism is a process of one-shot mechanisms (one mechanism after each outcome history), each of which is a mechanism with simultaneous moves and observed actions. (iii) The definition of implementability is that there is a subgame-perfect Nash equilibrium such that the social objective is fulfilled *at every period, after every outcome history*, as a unique equilibrium outcome process. When it is so we say that the SCF is (fully) *one-step-ahead implementable.*

We assume, as in Athey and Segal (2013), that the SCF is a (complete) contingent plan: In every period, after every history, it needs to specify a socially desirable outcome.

Under our notion of implementation, we provide two necessary conditions. The first and key necessary condition is what we call *folding* condition, which says that the SCF at a given period should depend only on agents' preferences over current social outcomes, which are induced by means of backward induction. Its role is to transforms the SCF into a list of one-shot functions, one for each period, so that period-t function maps every period-t environment into a period-t socially desirable outcome. Then, we show that every period-t function has to be Maskin monotonic (Maskin, 1999), after every history. This requirement is named *one-step-ahead Maskin monotonicity.*

Furthermore, if every period-t function satisfies unanimity and weak no veto power, we show that the necessary conditions are also sufficient. The construction of the process of one-shot mechanisms is simple: In each period t, after every history, just run the Maskin mechanism over period-t outcomes, in which each agent reports a profile of period-t preferences as well as a tie-breaking information.

Finally, we provide an account of welfare implications of our sufficiency result in the context of trading decisions and voting problems.

Firstly, we consider a borrowing-lending model with no liquidity constraints, in which agents trade in spot markets and transfer wealth between any two periods by borrowing and lending. In this set-up, intertemporal pecuniary externalities arise because trades in the current period change the spot price of the next period, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every agent's induced preference concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other agents. Under such a pecuniary externality, we show that the standard dynamic competitive equilibrium solution is not one-step-ahead implementable because it fails to satisfy the folding condition. We have also identified preference domains - which involve no pecuniary externalities - for which the dynamic competitive equilibrium solution is definable and one-step-ahead implementable.

[3] We may consider a set of such social choice functions in order to accommodate with possibility of multiple equilibria, although we consider a single function here for simplicity.

Secondly, we consider a bi-dimensional policy space where an odd number of agents vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that each voter's type space is unidimensional, that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This dynamic resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). In this environment, we show that the simple majority solution, which selects the Condorcet winner in each voting stage, is one-step-ahead implementable. In this process, we explicitly state the conditions on the utility function of each voter that are needed for this SCF to be well-defined and show that this is the case. As established by De Donder et al (2012) for the case where there is a continuum of voters, the assumption that both dimensions are strategic complements, as well as the requirement that the induced utility of both dimensions is increasing in the type of the voter, are particularly important for guaranteeing the existence of a Condorcet winner in each voting stage.

1.1 Related Literature

The fundamental paper on implementation is thanks to Maskin (1999; circulated since 1977), who proves that any choice rule that can be Nash implemented satisfies a remarkably strong invariance condition, now widely referred to as Maskin monotonicity. Moreover, he shows that when the mechanism designer faces at least three agents, a choice rule is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto-power.

Since Maskin's result, economists have also been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities offered by approximate (as opposed to exact) implementation (Matsushima, 1988; Abreu and Sen, 1991), as well as by implementation in refinements of Nash equilibrium (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Herrero and Srivastava, 1992; Jackson, 1992) and by repeated implementation (Kalai and Ledyard, 1998; Lee and Sabourian, 2011; Mezzetti and Renou, 2017).

Moore and Rupullo (1988)'s result says that for any choice rule, one can design an extensive game form that yields unique implementation in subgame-perfect Nash equilibria. They find that the class of implementable choice rules is dramatically expanded by the use of extensive game forms. In contrast to them, we restrict the attention to the class of multi-stage mechanisms with observed actions, and (perhaps) more important we are interested in implementing a social contingent plan rather than sequences of social outcomes. In terms of results, we find that we cannot escape the limitations imposed by Maskin monotonicity.

The paper on dynamic implementation, which is closest to ours, in particular because it allows for non-separable preferences and outcomes are chosen on a stage-by-stage basis, is Penta (2015). This author considers belief-free settings with incomplete information and restricts the analysis to direct mechanisms, but in many other respects, his environment is similar to ours (including resorting to a backwards procedure and identifying a property that is reminiscent of our necessary conditions).

Further, our dynamic problems contrast with the repeated implementation problems studied by Lee and Sabourian (2011) and Mezzetti and Renou (2017), in which agents' period-t preferences are time-separable and they change randomly from one period to the next one. Indeed, in our setup, the evolution of agents' period-t preferences are established as an endogenous process, because they depend on past social decisions and on planners' future goals.

The endogenous evolution of agents' information is a common feature in the most recent literature on dynamic mechanism design (see, e.g., Bergemann and Välimäki, 2010; and Pavan et al., 2014), though this literature maintains the assumption of separability of preferences and focuses on partial implementation.

Finally, our implementation problems also contrast with the static implementation problems studied by Hayashi and Lombardi (2017), in which planners solves their implementation problems simultaneously and do not communicate with each other. Indeed, in our setup, a period-t planner observes the outcome history and this history affects his implementation problem.

The remainder of the paper is organized as follows. Section 2 sets out the theoretical framework and outlines the basic implementation model. Section 3 presents our necessary and sufficient conditions. Section 4 covers one-step-ahead implementable SCFs in the context of trading and voting problems. Section 5 concludes. Appendix includes proofs not in the main body.

2. Basic framework

Let us imagine that a set of agents indexed by $i \in \mathcal{I} \equiv \{1, \cdots, I\}$ have to decide what outcome is best in each time period/stage indexed by $t \in \mathcal{T} \equiv \{1, 2, \cdots, T\}$. Let us denote the universal set of period-t outcomes by X^t, with x^t as a typical outcome. Thus, the universal set of outcome paths available to agents is the space:

$$\mathcal{X} \subseteq \prod_{t \in \mathcal{T}} X^t,$$

with x as a typical outcome path. The t-head x^{-t} is obtained from the path $x \in \mathcal{X}$ by omitting the last t components, that is, $x^{-t} \equiv (x^1, \cdots, x^{t-1})$, the t-tail is obtained from x by omitting the first $t-1$ components, that is, $x^{+t} \equiv \left(x^t, \cdots, x^T\right)$, and we identify (x^{-t}, x^{+t}) with x. The same notational convention will be followed for any profile of outcomes. We will refer to the t-head x^{-t} as the past outcome history x^{-t}.

The feasible set of period-$t+1$ outcomes available to agents depends upon past outcome history $x^{-(t+1)}$, that is, $X^{t+1}\left(x^{-(t+1)}\right) \subseteq X^{t+1}$ for every period $t \neq T$.

We write $\mathcal{F}^t$ for the collection of functions defined as follows:

$$\mathcal{F}^t \equiv \left\{f^t | f^t : \mathcal{X}^{-t} \to X^t \text{ such that } f^t\left[x^{-t}\right] \in X^t\left(x^{-t}\right)\right\}, \quad \text{for all } t \neq 1.$$

We also write $\mathcal{F}$ for the product space $X^1 \times \mathcal{F}^2 \times \cdots \times \mathcal{F}^T$.

The information held by the agents is summarized in the concept of a state, which is a complete description of the variable characterizing the environment. Write Θ for the domain of possible states, with θ as a typical state. For every period $t \geq 2$, the description of the variable characterizing the environment after the outcome history x^{-t} is denoted by $\theta|x^{-t}$. Moreover, for every $t \geq 2$ we write $\theta|x^{-t}, x^{+(t+1)}$ for a complete description of the variable characterizing the environment in period t after the outcome history x^{-t} and the future sure outcome path $x^{+(t+1)}$.

In the usual fashion, agent i's preferences in state θ are given by a complete and transitive binary relation, subsequently an ordering, $R_i(\theta)$ of elements of $\mathcal{X}$. The corresponding strict and indifference relations are denoted by $P_i(\theta)$ and $I_i(\theta)$, respectively. The statement $xR_i(\theta)\,y$ means that agent i judges x to be at least as good as y. The statement $xP_i(\theta)\,y$ means that agent i judges x to be better than y. Finally, the statement $xI_i(\theta)\,y$ means that agent i judges x and y as equally good, that is, she is indifferent between them.

2.1 Implementation model

Dynamic social objectives

The goal of the central designer is to implement a social choice function (SCF) $f : \Theta \to \mathcal{F}$ that assigns to each state θ a dynamic "socially optimal" process

$$f[\theta] = \left(f^1[\theta], f^2[\theta|\cdot], \cdots, f^T[\theta|\cdot]\right),$$

where:

- $f^1[\theta] \in X^1$ is the period-1 socially optimal outcome and
- $f^t[\theta|\cdot] \in \mathcal{F}^t$ is the period-t socially optimal process that selects the socially optimal outcome $f^t[\theta|x^{-t}]$ in period $t \geq 2$ at the state θ after the past outcome history $x^{-t} \in \mathcal{X}^{-t}$.

To save writing, for every period $t \neq 1$ and every past outcome history x^{-t}, we write $f^{+t}[\theta|x^{-t}]$ for the t-tail path of socially optimal outcomes in state θ that follows the past outcome history x^{-t}, whose period-τ element is the value of the composition $f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t$ at $\theta|x^{-t}$; that is:

$$f^{+t}\left[\theta|x^{-t}\right] \equiv \left(f^\tau\left[\theta|x^{-t}\right]\right)_{\tau \geq t}$$

where $f^\tau[\theta|x^{-t}] \equiv (f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t)[\theta|x^{-t}]$ for every period $\tau \geq t$. The image or range of the period-t function f^t of the SCF f at the past outcome history x^{-t} is the set:

$$f^t\left[\Theta|x^{-t}\right] \equiv \left\{f^t\left[\theta|x^{-t}\right]|\theta \in \Theta\right\}, \quad \text{for every } x^{-t} \in \mathcal{X}^{-t} \text{ with } t \neq 1.$$

The image or range of the period-1 function f^1 of the SCF f is the set $f^1[\Theta] \equiv \{f^1[\theta]\,|\theta \in \Theta\}$.

Dynamic mechanism

The central designer delegates the choice to agents according to a process of one-period mechanisms (or game forms) with observed actions and simultaneous moves and then execute that choice. In other words, we assume that the actions of every agent are perfectly monitored by every other agent as well as that every agent chooses an action in period t without knowing the period t action of any other agent.

More formally, in the first period all agents $i \in \mathcal{I}$ choose actions from nonempty choice sets $A_i\left(h^1\right)$, where $h^1 \equiv \varnothing$ denotes the initial history. Thus, the period-1 action space is the product space:

$$A\left(h^1\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^1\right),$$

with $a\left(h^1\right) \equiv \left(a_1^1\left(h^1\right), \cdots, a_I^1\left(h^1\right)\right)$ as a typical period-1 action profile.

In the second period, agents know the history $h^2 \equiv a^1$, and the actions that every agent $i \in \mathcal{I}$ has available in period 2 depends on what has happened previously. Then, let $A_i\left(h^2\right)$ denote the period-2 nonempty action space of agent i when the history is h^2 and let $A\left(h^2\right)$ denote the corresponding period-2 nonempty action space, which is defined by:

$$A\left(h^2\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^2\right),$$

with $a\left(h^2\right) \equiv \left(a_1\left(h^2\right), \cdots, a_I\left(h^2\right)\right)$ as a typical period-2 action profile.

Continuing iteratively, we can define h^t, the (nontrivial) history at the beginning of period $t > 1$, to be the list of $t-1$ action profiles,

$$h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right),$$

identifying actions played by agents in periods 1 through $t-1$. We let $A_i\left(h^t\right)$ be agent i's nonempty action set in period t when the history is h^t and let $A\left(h^t\right)$ be the corresponding period-t action space, which is defined by:

$$A\left(h^t\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(h^t\right),$$

with $a\left(h^t\right) \equiv \left(a_1\left(h^t\right), \cdots, a_I\left(h^t\right)\right)$ as a typical profile of actions.

We assume that in each period t, every agent knows the history h^t, this history is common knowledge at the beginning of period t, and that every agent $i \in I$ chooses an action from the action set $A_i\left(h^t\right)$. We also assume that in each period t, all agents $i \in I$ choose actions simultaneously.

We let H^t be the set of all period-t histories, where we define H^1 to be the null set, and let

$$H \equiv \bigcup_{t \in \mathcal{T}} H^t$$

be the set of all possible histories.

For any nontrivial history $h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right) \in H$, define a subhistory of h^t to be a sequence of the form $\left(a^1, \cdots, a^m\right)$ with $1 \leq m \leq t-1$, and the trivial history consisting of no actions is denoted by $\varnothing$.

The delegation to agents is made by means of a dynamic mechanism $\Gamma \equiv \left(\mathcal{I}, H, A\left(H\right), g\right)$, where H is the set of all possible histories, $A\left(H\right)$ is the set of all profiles of actions available to agents, defined by

$$A(H) \equiv \bigcup_{h \in H} A(h),$$

and $g \equiv \left(g^1, \cdots, g^T\right)$ is a sequence of outcome functions, one for each period $t \in \mathcal{T}$, with the property that: a) the outcome function g^1 assigns to period-1 action profile $a(h^1) \in A(h^1)$ a unique outcome in X^1, and b) for every period $t \neq 1$ and every nontrivial history $h^t \equiv (a^1, a^2, \cdots, a^{t-1}) \in H^t$, the outcome function g^t assigns to each period-t action profile $a(h^t) \in A(h^t)$ a unique outcome in $X^t(g^{-t}(h^t))$.

The submechanism of a dynamic mechanism Γ that follows the history h^t is the dynamic mechanism

$$\Gamma\left(h^t\right) \equiv \left(\mathcal{I}, H|h^t, A\left(H|h^t\right), g^{+t}\right),$$

where $H|h^t$ is the set of histories for which h^t is a subhistory for every $h \in H|h^t$,

$$A\left(H|h^t\right) \equiv \bigcup_{h \in H|h^t} A(h)$$

is the set of all profiles of actions available to agents from period t to period T, and g^{+t} is t-tail of the sequence g that begins with period t after the history h^t such that for every $h^T \equiv \left(a^1, \cdots, a^{T-1}\right) \in H^T|h^t$ and every $a\left(h^T\right) \in A\left(h^T\right)$ it holds that $g\left(h^T, a\left(h^T\right)\right) = \left(g^{-t}\left(h^T, a\left(h^T\right)\right), g^{+t}\left(h^T, a\left(h^T\right)\right)\right)$.

One-step-ahead implementation

A dynamic mechanism Γ and a state θ induce a dynamic game (Γ, θ) (with observed actions and simultaneous moves). The subgame of the dynamic game (Γ, θ) that follows the history $h^t \in H$ is the dynamic game $(\Gamma(h^t), \theta)$.

Let $A_i \equiv \bigcup_{h \in H} A_i(h)$ be the set of all actions for agent $i \in \mathcal{I}$. A (pure) strategy for agent i is a map $s_i : H \to A_i$ with $s_i(h) \in A_i(h)$ for every history $h \in H$. Individual i's space of strategies, S_i, is simply the space of all such s_i.

A strategy profile $s \equiv (s_1, \cdots, s_I)$ is a list of strategies, one for each agent $i \in \mathcal{I}$. The strategy profile s_{-i} is obtained from s by omitting the ith component, that is, $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$, and we identify (s_i, s_{-i}) with s.

For any strategy s_i of agent i and any history h^t in the dynamic mechanism Γ, the strategy that s_i induces in the dynamic subgame $(\Gamma(h^t), \theta)$ is denoted by $s_i|h^t$. Individual i's space of strategies that follows history h^t is denoted by $S_i|h^t$. The period-t strategy of agent i is sometimes denoted by s_i^t.

For every dynamic game (Γ, θ), the strategy profile s^* is a Nash equilibrium of (Γ, θ) if for every agent $i \in \mathcal{I}$ it holds that:

$$g\left(s_i^*, s_{-i}^*\right) R_i(\theta)\, g\left(s_i, s_{-i}^*\right) \text{ for every } s_i \in S_i.$$

Let $NE(\Gamma, \theta)$ denote the set of Nash equilibrium strategy profiles of (Γ, θ).

Moreover, for every dynamic game (Γ, θ) and every nontrivial history $h^t \in H$, the strategy profile $s^*|h^t$ is a Nash equilibrium of $(\Gamma(h^t), \theta)$ if for every agent $i \in \mathcal{I}$ and a given past outcome history $x^{-t} \in \mathcal{X}^{-t}$ it holds that:

$$\left(x^{-t}, g^{+t}\left(s_i^*|h^t, s_{-i}^*|h^t\right)\right) R_i(\theta) \left(x^{-t}, g^{+t}\left(s_i|h^t, s_{-i}^*|h^t\right)\right) \text{ for every } s_i|h^t \in S_i|h^t.$$

Let $NE(\Gamma(h^t), \theta)$ denote the set of Nash equilibrium strategy profiles of $(\Gamma(h^t), \theta)$.

A strategy profile s^* is a *subgame perfect equilibrium* (SPE) of a dynamic game (Γ, θ) if it holds that:

$$s^*|h^t \in NE\left(\Gamma\left(h^t\right), \theta\right), \qquad \text{for every history } h^t \in H.$$

Let $SPE(\Gamma, \theta)$ denote the set of SPE strategy profiles of (Γ, θ), with s^θ as a typical element.

Definition 1 A dynamic mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ implements the SCF $f : \Theta \to \mathcal{F}$ in SPE if for every $\theta \in \Theta$,

$$f^1[\theta] = g^1\left(s^\theta\left(h^1\right)\right), \text{ and}$$

$$f^t\left[\theta|g^{-t}\left(h^t\right)\right] = g^t\left(s^\theta\left(h^t\right)\right), \quad \text{for every } h^t \in H^t \text{ with } t \neq 1,$$

if and only if

$$s^\theta \in SPE(\Gamma, \theta)$$

If such a mechanism exists, the SCF f is said to be *one-step-ahead implementable.*

3. Necessary and sufficient conditions

3.1 Folding

In this section, we first propose a property, *folding*, and show that this is a necessary condition for one-step-ahead implementation. While this property is heavy in notation, its idea is simple. This property reduces the dynamic implementation problem into a process of "step-by-step" implementation problems, one for each period, where the planner takes only the agents' preferences induced over current social outcomes into account.

This necessary condition is derived by using the approach developed by Moore and Repullo (1990) and thus it is stated in terms of the existence of certain sets. These sets are denoted by $\mathcal{Y}^{-t}$, Y^1 and $Y^t(y^{-t})$ and represent respectively the set of feasible past outcome histories up to period $t \neq 1$, the set of period-1 attainable outcomes and the set of period-t attainable outcomes after the past outcome history y^{-t}. Moreover, the condition consists of three parts: the first part characterises the period-T implementation problem, the second one relates to the implementation problem of period $t \neq 1, T$ and the third one relates to the period-1 implementation problem.

Solving backward, for any feasible past outcome history y^{-T}, the *period-T induced ordering* of agent i in state θ at y^{-T}, that is, at $\theta|y^{-T}$, denoted by $R_i\left[\theta|y^{-T}\right]$, is equal to:

$$y^T R_i\left[\theta|y^{-T}\right] z^T \iff \left(y^{-T}, y^T\right) R_i(\theta) \left(y^{-T}, z^T\right), \quad \text{for every } y^T, z^T \in Y^T\left(y^{-T}\right). \qquad (1)$$

We denote by $R\left[\theta|y^{-T}\right]$ the profile of period-T induced orderings at $\theta|y^{-T}$ and by

$\mathcal{D}\left[\Theta|y^{-T}\right]$ the period-T domain of induced orderings at $\Theta|y^{-T}$, that is:

$$\mathcal{D}\left[\Theta|y^{-T}\right] \equiv \left\{R\left[\theta|y^{-T}\right]|\theta \in \Theta\right\}. \tag{2}$$

Therefore, the first part of the condition can be formulated as follows:

(i) The preference domain $\mathcal{D}\left[\Theta|y^{-T}\right]$ is not empty, and there is a period-T function φ^T : $\mathcal{D}\left[\Theta|y^{-T}\right] \to Y^T\left(y^{-T}\right)$ such that:

$$\varphi^T\left(R\left[\theta|y^{-T}\right]\right) = f^T\left[\theta|y^{-T}\right], \quad \text{for every } \theta \in \Theta. \tag{3}$$

To introduce the second part of the condition, let us suppose that in our way back to period 1 we have reached period $t \neq 1, T$ and that y^{-t} is a feasible past outcome history. Given that in our framework rationality is common knowledge between the players and given that the objective of the planner is to implement a dynamic social choice process prescribed by the SCF f, every player will "look ahead" and a period-t outcome y^t will be evaluated at the past outcome history y^{-t} as well as at the future sure outcome path $f^{+(t+1)}$ prescribed by the SCF in response to the outcome history path (y^{-t}, y^t). On this basis, the *period-t induced ordering* of agent i in state θ at the past outcome history y^{-t} and at the future sure outcome path prescribed by the social process $f^{+(t+1)}$, that is, at $\theta|y^{-t}, f^{+(t+1)}$, denoted by $R_i\left[\theta|y^{-t}, f^{+(t+1)}\right]$, is equal to:

$$\begin{aligned} &y^t R_i\left[\theta|y^{-t}, f^{+(t+1)}\right] z^t \iff \\ &\quad \left(y^{-t}, y^t, f^{+(t+1)}\left[\theta|\left(y^{-t}, y^t\right)\right]\right) R_i(\theta) \left(y^{-t}, z^t, f^{+(t+1)}\left[\theta|\left(y^{-t}, z^t\right)\right]\right), \end{aligned} \tag{4}$$

for every $y^t, z^t \in Y^t\left(y^{-t}\right)$.

Let us denote by $R\left[\theta|y^{-t}, f^{+(t+1)}\right]$ the profile of period-t induced orderings at $\theta|y^{-t}, f^{+(t+1)}$ for $t \neq 1, T$ and by $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ the period-t domain of induced orderings at $\Theta|y^{-t}, f^{+(t+1)}$, that is:

$$\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right] \equiv \left\{R\left[\theta|y^{-t}, f^{+(t+1)}\right]|\theta \in \Theta\right\}. \tag{5}$$

Therefore, as for the first part of the condition, the second part can be stated as follows:

(ii) The preference domain $\mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$ is not empty, and there is a period-t function $\varphi^t : \mathcal{D}\left[\Theta|y^{-t}, f^{+(t+1)}\right] \to Y^t\left(y^{-t}\right)$ such that:

$$\varphi^t\left(R\left[\theta|y^{-t}, f^{+(t+1)}\right]\right) = f^t\left[\theta|y^{-t}\right], \quad \text{for every } \theta \in \Theta. \tag{6}$$

Reasoning like that used in the preceding paragraphs, the *period-1 induced ordering* of agent i in state θ at the outcome path prescribed by the social process f^{+2}, that is, at $\theta|f^{+2}$, denoted by $R_i\left[\theta|f^{+2}\right]$, is equal to:

$$y^1 R_i\left[\theta|f^{+2}\right] z^1 \iff \left(y^1, f^{+2}\left[\theta|y^1\right]\right) R_i(\theta) \left(z^1, f^{+2}\left[\theta|z^1\right]\right), \quad \text{for every } y^1, z^1 \in Y^1. \tag{7}$$

Denoting the profile of period-1 induced orderings at $\theta|f^{+2}$ by $R\left[\theta|f^{+2}\right]$ and defining the period-1 domain of induced orderings at $\Theta|f^{+2}$ by:

$$\mathcal{D}\left[\Theta|f^{+2}\right] \equiv \left\{R\left[\theta|f^{+2}\right]|\theta \in \Theta\right\}, \tag{8}$$

the third part of folding can be stated as follows:

(iii) The preference domain $\mathcal{D}[\Theta|f^{+2}]$ is not empty, and there is a period-1 function $\varphi^1 : \mathcal{D}[\Theta|f^{+2}] \to Y^1$ such that:

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) = f^1[\theta], \quad \text{for every } \theta \in \Theta. \tag{9}$$

In summary, if the SCF f is one-step-ahead implementable, then the following condition must be satisfied:

Definition 2 The SCF $f : \Theta \to \mathcal{F}$ satisfies the *folding condition* if there is a collection of spaces of sequences of past outcomes $\{\mathcal{Y}^{-t}\}_{t\in\mathcal{T}\setminus\{1\}}$ and if there is a period-1 outcome space $Y^1 \equiv \mathcal{Y}^{-2}$ and there is a collection of period-t outcome spaces $\left\{\{Y^t(y^{-t})\}_{y^{-t}\in\mathcal{Y}^{-t}}\right\}_{t\in\mathcal{T}\setminus\{1\}}$ such that:

- $f^1[\Theta] \subseteq Y^1$ and $f^t[\Theta|y^{-t}] \subseteq Y^t(y^{-t})$ for every $t \neq 1$;
- for every $t \neq 1$, it holds that

$$y^{-t} \in \mathcal{Y}^{-t} \iff y^1 \in Y^1 \text{ and } y^\tau \in Y^\tau\left(y^{-\tau}\right) \text{ for every } 2 \leq \tau \leq t;$$

- part *(i)* is satisfied for every $y^{-T} \in \mathcal{Y}^{-T}$;
- part *(ii)* is satisfied for every $y^{-t} \in \mathcal{Y}^{-t}$ with $t \neq 1, T$;
- part *(iii)* is satisfied.

A SCF satisfying the above condition is said to be a *folding SCF*. Our first main result can thus be stated as follows:

Theorem 1 If $I \geq 2$ and the SCF $f : \Theta \to \mathcal{F}$ is one-step-ahead implementable, then it satisfies the folding condition.

Proof. See Appendix. ■

This is not an obvious condition. In intertemporal environments there may be various different reasons for sending an identical message in a single period. For example, consider that an agent sends a message telling that he prefers high tax rate in the current period. It may be because he purely prefers to have high tax in the current period, or it may be because he desires the tax rate to be chosen in the next period following the high current tax rate, and so on. The folding condition says that the planner does not distinguish between those reasons, and does not need to question why either.

3.2 One-step-ahead Maskin monotonicity

A condition that is central to the Nash implementation thanks to Maskin (1999) is an invariance condition, now widely referred to as Maskin monotonicity. This condition says that if an outcome x is socially optimal at the state θ and this x does not strictly fall in preference for anyone when the state is changed to θ', then x must remain a socially optimal outcome at θ'. An equivalent statement of Maskin monotonicity follows the reasoning that if x is socially optimal at θ but not socially optimal at θ', then the outcome x must have fallen strictly in someone's ordering at the state θ' in order to break the Nash equilibrium via some deviation. Therefore, there must exist some (outcome-)preference reversal if a Nash equilibrium strategy profile at θ is to be broken at θ'. Let us formalize that condition as follows: For any state θ and any agent i and any outcome $x \in X$, the weak lower contour set of $R_i(\theta)$ at x is defined by $L(x, R_i(\theta)) \equiv \{y \in X | x R_i(\theta) y\}$. Therefore:

Definition 3 The SCF $F : \Theta \to X$ is *Maskin monotonic* provided that for all $x \in X$ and all $\bar{\theta}, \theta \in \Theta$, if $L(f(\bar{\theta}), R_i(\bar{\theta})) \subseteq L(f(\bar{\theta}), R_i(\theta))$ for every $i \in \mathcal{I}$, then $f(\bar{\theta}) = f(\theta)$.

We basically require an adaptation of Maskin monotonicity to each implementation problem. In other words, one-step-ahead Maskin monotonicity requires that every period-t social choice function φ^t that results from the folding of the SCF is Maskin monotonic. Therefore, the condition of one-step-ahead Maskin monotonicity can be stated as follows:

Definition 4 A folding SCF $f : \Theta \to \mathcal{F}$ is *one-step-ahead Maskin monotonic* provided that: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta | y^{-T}\right]$ is Maskin monotonic for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta | y^{-t}, f^{+(t+1)}\right]$ is Maskin monotonic for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}\left[\Theta | f^{+2}\right]$ is Maskin monotonic.

Our second main result is that only one-step-ahead Maskin monotonic SCFs are one-step-ahead implementable.

Theorem 2 If $I \geq 2$ and the SCF $f : \Theta \to \mathcal{F}$ is one-step-ahead implementable, then it is one-step-ahead Maskin monotonic.

Proof. See Appendix. ■

3.3 The characterization theorem

In the abstract Arrovian domain, the condition of no veto-power says that if an outcome is at the top of the preferences of all agents but possibly one, then it should be chosen irrespective of the preferences of the remaining agent: that agent cannot veto it. The condition of no veto-power implies two well-known conditions: *unanimity* and *weak no veto-power*. Unanimity states that if an outcome is at the top of the preferences of all agents, then that outcome should be selected by the SCF. Weak no veto-power states that if an outcome x is socially optimal at the state $\bar{\theta}$ and if the state changes from $\bar{\theta}$ to θ in a way that under the new state an outcome y that was no better than x at $R_i(\bar{\theta})$ for some agent i is weakly preferred to all outcomes in the weak lower contour set of $R_i(\bar{\theta})$ at x according to the ordering $R_i(\theta)$ and this y is maximal for every other agent j in the set X according to $R_j(\theta)$, then this y should be socially optimal at θ. These properties can be stated as follows for an abstract outcome space X:

Definition 5 The SCF $F : \Theta \to X$ satisfies *unanimity* provided that for all $\theta \in \Theta$ and all $x \in X$ if $xR_i(\theta)y$ for all $i \in \mathcal{I}$ and all $y \in X$, then $x = F(\theta)$. A SCF that satisfies this property is said to be a unanimous SCF.

Definition 6 A SCF $F : \Theta \to X$ satisfies *weak no veto-power* provided that for every $\bar{\theta}, \theta \in \Theta$ if $y \in L\left(f\left(\bar{\theta}\right), R_i\left(\bar{\theta}\right)\right) \subseteq L(y, R_i(\theta))$ for some $i \in \mathcal{I}$ and $X \subseteq L(y, R_j(\theta))$ for every $j \in \mathcal{I} \setminus \{i\}$, then $f(\theta) = y$.

As a part of sufficiency, we require an adaptation of the above definitions to each period-t implementation problem. In other words, one-step-ahead unanimity requires that each of period-t social function φ^t defined over period-t domain of induced orderings is unanimous, whereas one-step-ahead weak no veto-power requires that each φ^t satisfies weak no veto-power. The conditions can be stated as follows:

Definition 7 A folding SCF $f : \Theta \to \mathcal{F}$ satisfies *one-step-ahead unanimity* provided that the following requirements hold: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta | y^{-T}\right]$ is unanimous for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta | y^{-t}, f^{+(t+1)}\right]$ is unanimous for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}\left[\Theta | f^{+2}\right]$ is unanimous.

Definition 8 A folding SCF $f : \Theta \to \mathcal{F}$ satisfies *one-step-ahead weak no veto-power* provided that the following requirements hold: *(i)* the period-T function φ^T over $\mathcal{D}\left[\Theta | y^{-T}\right]$ satisfies weak no veto-power for every $y^{-T} \in \mathcal{Y}^{-T}$; *(ii)* for every $t \neq 1, T$, the period-t function φ^t over $\mathcal{D}\left[\Theta | y^{-t}, f^{+(t+1)}\right]$ satisfies weak no veto-power for every $y^{-t} \in \mathcal{Y}^{-t}$; *(iii)* the period-1 function φ^1 over $\mathcal{D}\left[\Theta | f^{+2}\right]$ satisfies weak no veto-power.

Our characterization of one-step-ahead implementable SCFs can thus be stated as follows:

Theorem 3 If $I \geq 3$ and the SCF $f : \Theta \to \mathcal{F}$ satisfies the folding condition and one-step-ahead Maskin monotonicity and if the SCF satisfies one-step-ahead weak no veto-power as well as one-step-ahead unanimity, then it is one-step-ahead implementable.

Proof. See Appendix. ■

4. Implications

4.1 Impossibility of implementing the dynamic competitive solution

In this section, we investigate whether the trading rule as considered in the dynamic general equilibrium framework is indeed one-step-ahead implementable.

When it is literally understood, the concept of Arrow-Debreu-McKenzie (ADM) (Arrow and Debreu, 1954; McKenzie,1954) equilibrium says that all the agents meet on the first day of their life and write down a contract on all the deliveries of consumption contingent on every date-event, and simply commit to it. A more realistic description of trading over time is by Radner (1972, 1982), which considers that at each period agents can trade only between current consumption and assets to be carried over to the next period. To our knowledge, however, the Radner-type model has not been given a strategic foundation. In the Radner model prices are defined only for *on-path* situation and it is left unclear what prices should be formed in *off-path* situations, while a strategic outcome function in a dynamic environment must specify prices and allocations even at off-path histories. In fact, as far as the markets are sequentially complete ADM equilibrium and Radner equilibrium are equivalent (Arrow, 1964). This means that from strategic viewpoints the Radner model cannot escape the problem which the ADM model has. The competitive models are silent about what prices and allocations should be formed after the society makes mistake.

Strategic implementation of competitive solutions in general involves a strange story: each agent is supposed to behave as a price-taker, despite he is aware that message he sends may affect the market price. In the static setting, these apparently contradicting natures can be made compatible by making the mechanism nicely so that agents face a kind of coordination game in which they are induced to agree on prices in equilibrium. In fact, the (feasibility-constrained version of) ADM solution is Nash-implementable.

Being a price-taker is harder in dynamic environments, however, when social decision and execution can be made only in a sequential manner. It requires that every agent perceives that he cannot affect spot price/interest rate at any period, in particular that the amount of asset to carry over to the future does not affect the spot prices/interest rates in the future, despite he is aware that messages he sends may affect the market price in both the current period and the future periods, and that equilibrium prices and allocations in the future periods are a function of *whole allocations* in the current period including his own.

Below we explain the nature of the problem and see whether the Radner-type solution can clear this bar.

For the sake of convenience, we assume that there are only *three* consumption periods (CPs), and so *two* trading periods (TPs), and that there is one perfectly divisible commodity in each CP. In TP1 agents transfer consumption between CP1 and CP2, and in TP2 they transfer consumption between CP2 and CP3.

In TP1, agents sell/buy consumption in CP1 and buy/sell consumption in CP2. In TP2, agents sell/buy consumption in CP2 and buy/sell consumption in CP3. Let q^1 be the TP1 spot price, the relative price of CP2 consumption for CP1 consumption, and q^2 be the TP2 spot price, the relative price of CP3 consumption for CP2 consumption.

Each agent i is endowed with an amount ω_i^t of the commodity in CPt. The total endowment of the commodity in CPt is denoted by ω^t. Agent i's consumption set is $\mathbb{R}^3_+$, and her consumption in CPt is denoted by c_i^t. In state θ, this agent has preference ordering $R_i(\theta)$ over consumption sequences in her consumption set. Endowments are given once and for all, and therefore an *economy* is described by a state θ.

The domain assumption is that at each economy $\theta \in \Theta$ agent i's preference ordering $\succcurlyeq_i^\theta$ is represented by an additively separable utility function

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = v_i^1(\theta, c_i^1) + v_i^2(\theta, c_i^2) + v_i^3(\theta, c_i^3).$$

This guarantees that all consumption goods in CP1, CP2 CP3 are gross-substitutes of each other and the ADM and Radner equilibrium is unique.

We describe feasible allocations by using net trade vectors. Let

$$H = \left\{ z \in \mathbb{R}^I | \sum_{i \in \mathcal{I}} z_i = 0 \right\},$$

which is the set of closed net trades. Thus, the set of closed net trade vectors for TPt can be defined by

$$Z^t = H^t \times H^{t+1}, \qquad \text{for } t = 1, 2.$$

A TPt net trade allocation is thus a vector $z^t = (z^{tt}, z^{tt+1})$ in Z^t, where the ith element z_i^{tt} of z^{tt} denotes agent i's net trade of consumption in CPt, and where the ith element z_i^{tt+1} of z^{tt+1} denotes agent i's net trade of consumption in CP$(t+1)$.

The set of feasible net trade allocations over the two trading periods is denoted by Z and defined by

$$Z = \left\{ (z^1, z^2) \in Z^1 \times Z^2 | \omega_i^1 + z_i^{11} \geq 0,\ \omega_i^2 + z_i^{12} + z_i^{22} \geq 0, \omega_i^3 + z_i^{23} \geq 0,\ \forall i \in \mathcal{I} \right\}.$$

The set of feasible TP1 net trade allocations is given by

$$\bar{Z}^1 = \{ z^1 \in Z^1 | (z^1, z^2) \in Z \text{ for some } z^2 \in Z^2 \},$$

while the set of TP2 net trade allocation, conditional on z^1, is given by

$$\bar{Z}^2(z^1) = \{ z^2 \in Z^2 | (z^1, z^2) \in Z \}, \quad \text{for all } z^1 \in \bar{Z}^1.$$

In economy $\theta \in \Theta$, agent i's preference ordering $\succcurlyeq_i^\theta$ over consumption sequences induces a preference ordering $R_i(\theta)$ over the set of feasible net trade allocations Z in the natural way: for all $z, \hat{z} \in Z$,

$$\begin{aligned} z R_i(\theta) \hat{z} \quad \Longleftrightarrow \quad & U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}) \geq \\ & \qquad U_i(\theta, \omega_i^1 + \hat{z}_i^{11}, \omega_i^2 + \hat{z}_i^{12} + \hat{z}_i^{22}, \omega_i^3 + \hat{z}_i^{23}). \end{aligned}$$

Though the preference ordering $\succcurlyeq_i^\theta$ exhibits separability over consumption sequences, the derived preference ordering $R_i(\theta)$ over Z is typically non-separable since consumption in CP2 depends on net trades in both TP1 and TP2.

We provide the definition of competitive equilibrium backward. The definition of equilibrium when we start from TP2 is straightforward.[4]

Definition 9 For every economy $\theta \in \Theta$ and every $z^1 \in \bar{Z}^1$, the net trade allocation

[4] Note that this is a feasibility-constrained version. As it is known that the ADM solution fails to satisfy Maskin monotonicity when it results in boundary allocations, and it is necessary to modify the solution by truncating each agent's consumption set by the set of feasible allocations. Here each individual's admissible set of trades is truncated by $\bar{Z}^2(z^1)$, although it does not matter when we can restrict attention to interior allocations.

$f^2[\theta|z^1] \in \bar{Z}^2(z^1)$ constitutes a *TP2 competitive net trade allocation*, conditional on z^1, if there is a TP2 spot price $q^2[\theta|z^1]$ such that for every agent i this allocation $f^2[\theta|z^1]$ solves the following problem:

$$\underset{z^2 \in \bar{Z}^2(z^1)}{\text{Max}} \quad U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}), \qquad \text{subject to } z_i^{22} + q^2[\theta|z^1]z_i^{23} \leq 0. \quad (10)$$

Let $R_i^1[\theta, f^2]$ denote agent i's TP1 induced preference ordering over the set of feasible TP1 net trade allocations, $\bar{Z}^1$, and be defined by

$$\begin{aligned} x^1 R_i^1[\theta, f^2] y^1 \iff & \; U_i(\theta, \omega_i^1 + x_i^{11}, \omega_i^2 + x_i^{12} + f_i^{22}[\theta|x^1], \omega_i^3 + f_i^{23}[\theta|x^1]) \quad (11) \\ \geq & \; U_i(\theta, \omega_i^1 + y_i^{11}, \omega_i^2 + y_i^{12} + f_i^{22}[\theta|y^1], \omega_i^3 + f_i^{23}[\theta|y^1]), \qquad \text{for all } x^1, y^1 \in \bar{Z}^1. \end{aligned}$$

In contrast to static pure exchange economies where each agent's preferences are defined over her own net trade vectors, in sequential trading, each agent must have preferences over *whole* TP1 net trade allocations. This is due to the presence of *intertemporal pecuniary externalities*. Indeed, an outcome of the trading rule in TP2 depends on the net trade allocation assigned in TP1, because trading in TP1 affects the values of endowments in the next trading period. Moreover, the induced ordering $R_i^1[\theta, f^2]$ may be *non-convex*. In order for it to be a convex preference ordering, it is required that the TP2 function f^2 that maps every economy, conditional on past trades, into a TP2 net trade allocation be a concave function, but this requirement fails for any reasonable trading rule. As is known, although convexity is no more than a sufficient technical condition for things to work, it becomes extremely difficult to establish any reasonable solution once it is violated.

We may proceed in two ways. First, we can still *define* a concept of competitive equilibrium following the tradition of dynamic general equilibrium theory.

DEFINITION 10 For every economy $\theta \in \Theta$, a TP1 net trade allocation $f^1[\theta] \in \bar{Z}^1$ constitutes a *TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation profile $(f^1[\theta], f^2[\theta|f^1[\theta]])$ solves the following problem:

$$\underset{z \in Z}{\text{Max}} \quad U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23})$$

subject to

$$\begin{aligned} \text{(i)} \quad z_i^{11} + q^1[\theta]z_i^{12} &\leq 0 \\ \text{(ii)} \quad z_i^{22} + q^2[\theta|f^1[\theta]]z_i^{23} &\leq 0. \end{aligned}$$

This is consistent with the existing dynamic general equilibrium framework, in the sense that agents take the price *path* as given. Note that it assumes that each agent perceives that her saving choice does not affect either TP1 spot price $q^1[\theta]$ or TP2 spot price $q^2[\theta|f^1[\theta]]$, despite that in the next period the spot price $q^2[\theta|z^1]$ is affected by whole z^1 which includes his own trade vector z_i^1.

The *path* of consumptions given by this solution is equivalent to the ADM solution.[5]

[5]Note again that this is the feasibility-constrained version, while it does not matter when we can restrict

This solution is not one-step-ahead implementable, however. We prove this by means of an example.

CLAIM 1 Let $I \geq 2$. Then, the Radner solution, defined over Θ, does not satisfy the folding condition.

PROOF. Suppose that there are three agents, i, j and k. Assume that agents' intertemporal endowments are as follows:

$$\omega_i = (\omega_i^1, 0, 0),\ \omega_j = (0, \omega_j^2, 0) \text{ and } \omega_k = (0, 0, \omega_k^3),$$

where $\omega_i^1, \omega_j^2, \omega_k^3 > 1$.

Each economy $\theta \in \Theta = (0, 1]$ specifies a preference profile over consumption paths represented by:

$$\begin{aligned} U_i(\theta, c_i^1, c_i^2, c_i^3) &= c_i^1 + \theta \ln c_i^3 \\ U_j(\theta, c_j^1, c_j^2, c_j^3) &= \ln c_j^1 + c_j^2 \\ U_k(\theta, c_k^1, c_k^2, c_k^3) &= \ln c_k^2 + c_k^3. \end{aligned}$$

Then, the TP2 spot price equilibrium is given by:

$$q^2[\theta|x^1] = x_i^{12}, \qquad \text{for all } x^1 \in \bar{Z}^1,$$

and the TP2 competitive net trade allocation is given by:

$$\begin{aligned} f_i^{22}[\theta|x^1] &= -x_i^{12} \\ f_i^{23}[\theta|x^1] &= 1 \\ f_j^{22}[\theta|x^1] &= 0 \\ f_j^{23}[\theta|x^1] &= 0 \\ f_k^{22}[\theta|x^1] &= x_i^{12} \\ f_k^{23}[\theta|x^1] &= -1, \qquad \text{for all } x^1 \in \bar{Z}^1. \end{aligned}$$

The TP1 orderings over $\bar{Z}^1$ induced by TP2 competitive net trade allocations are represented respectively by:

$$\begin{aligned} U_i^1\left(\theta, x^1|f^2\right) &= \omega_i^1 + x_i^{11} \\ U_j^1\left(\theta, x^1|f^2\right) &= \ln x_j^{11} + \omega_j^2 + x_j^{12} \\ U_k^1\left(\theta, x^1|f^2\right) &= \ln x_i^{12} + \omega_k^3 - 1, \qquad \text{for all } x^1 \in \bar{Z}^1, \text{ for all } \theta \in \Theta. \end{aligned}$$

For every economy $\theta \in \Theta$, the TP1 equilibrium spot price is:

$$q^1[\theta] = \theta,$$

attention to interior allocations.

which results in the following TP2 equilibrium spot price:

$$q^2[\theta|f^1[\theta]] = 1,$$

and in the following competitive equilibrium net trade allocations:

$$\begin{aligned}
f_i^{11}[\theta] &= -\theta \\
f_i^{12}[\theta] &= 1 \\
f_i^{22}[\theta|f^1[\theta]] &= -1 \\
f_i^{23}[\theta|f^1[\theta]] &= 1 \\
f_j^{11}[\theta] &= \theta \\
f_j^{12}[\theta] &= -1 \\
f_j^{22}[\theta|f^1[\theta]] &= 0 \\
f_j^{23}[\theta|f^1[\theta]] &= 0 \\
f_k^{11}[\theta] &= 0 \\
f_k^{12}[\theta] &= 0 \\
f_k^{22}[\theta|f^1[\theta]] &= 1 \\
f_k^{23}[\theta|f^1[\theta]] &= -1.
\end{aligned}$$

We have found that $f^1[\theta] \neq f^1[\theta']$ for all $\theta,\theta' \in \Theta$ with $\theta \neq \theta'$, though TP1 reduced utility profiles are identical across economies in Θ, in violation of part (iii) of the folding condition. ■

When an agent reveals an intention to save more, there may be different reasons to do so. It may be because he is simply patient, or it may be because he wants to manipulate the Radner equilibrium outcome in the next period. The planner does not distinguish between those reasons. In particular, the above example is the case that *no* information is revealed to the planner after TP1.

While we can induce agents to behave as price-takers in the static setting, it is thus in general impossible to do that in the realistic dynamic setting in which social decision and execution can be done only in a sequential manner. The problem will disappear when there is a large number of traders, as each agent tends to be small and unable to manipulate through intertemporal pecuniary externalities. Then we would say that the dynamic general equilibrium model should be understood as such a limit model rather than an exact finite-person model.

The second way is find a domain in which an exact finite-person implementation is possible. It is the domain such that there are no intertemporal pecuniary externalities.[6]

Condition 1 For all $\theta \in \Theta$, the TP2 spot price $q^2[\theta|x^1]$ is constant in $x^1 \in \bar{Z}^1$.

Note that when the above condition is met, a TP2 competitive net trade vector assigned

[6]Such situation emerges also when constant returns to scale in intertemporal production prevails, since interest rate in such economy is constant.

to agent i depends only on her own past saving/borrowing behavior. For this reason, we write $f_i^{22}[\theta|z_i^{12}]$ and $f_i^{23}[\theta|z_i^{12}]$ for $f_i^{22}[\theta|z^1]$ and $f_i^{23}[\theta|z^1]$ respectively.

Here are examples of domains which satisfy Condition 1. In what follows, let us focus on economies where the quantity ω_i^t is strictly positive for every agent i and every consumption period $t = 1, 2, 3$.

Assumption 1 (Θ^1) Assume that aggregate endowment is constant over time; that is, $\omega^1 = \omega^2 = \omega^3$. Also, assume that the agents have identical discount factors, while they may exhibit different elasticities of intertemporal substitution. That is, for every economy $\theta \in \Theta^1$ it holds that $\omega^1 = \omega^2 = \omega^3$ and that there is (β^1, β^2) such that every i's preference over consumptions is represented in the form:

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = v_i(\theta, c_i^1) + \beta^1 v_i(\theta, c_i^2) + \beta^1\beta^2 v_i(\theta, c_i^3),$$

where:

- the sub-utility $v_i(\theta, \cdot)$ is twice continuously differentiable, strictly increasing and strictly concave over $\mathbb{R}_{++}$.
- the limit of the first derivative of the sub-utility $v_i(\theta, \cdot)$ is positive infinity as c_i^t approaches 0; that is, $\lim_{c_i^t \to 0} \frac{\partial v_i(\theta, c_i^t)}{\partial c_i^t} = \infty$.
- the limit of the first derivative of the sub-utility $v_i(\theta, \cdot)$ is zero as c_i^t approaches positive infinity; that is, $\lim_{c_i^t \to \infty} \frac{\partial v_i(\theta, c_i^t)}{\partial c_i^t} = 0$.
- the sub-utility $v_i(\theta, \cdot)$ satisfies the requirement that $-\left(\frac{\partial^2 v_i(\theta, c_i^t)}{\partial^2 c_i^t} c_i^t / \frac{\partial v_i(\theta, c_i^t)}{\partial c_i^t}\right) < 1$ for all $c_i^t \in \mathbb{R}_{++}$.

For this domain, we obtain that the TP2 competitive spot price, net trade allocations and consumption allocations prescribed for every $\theta \in \Theta^1$ are:

$$\begin{aligned}
q^2\left[\theta|z^1\right] &= \beta^2 \\
f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{\beta^2}{1+\beta^2} \cdot \left(z_i^{12} + \omega_i^2 - \omega_i^3\right) \\
f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{1}{1+\beta^2} \cdot \left(z_i^{12} + \omega_i^2 - \omega_i^3\right) \\
c_i^{*2}\left[\theta|z^1\right] &= c_i^{*3}\left[\theta|z^1\right] = \frac{z_i^{12} + \omega_i^2 + \beta^2\omega_i^3}{1+\beta^2}, \quad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1.
\end{aligned}$$

Note that period-1 reduced utility on $\bar{Z}^1$ is represented by:

$$U_i\left(\theta, z^1|f^2\right) = v_i\left(\theta, \omega_i^1 + z_i^{11}\right) + \beta^1\left(1+\beta^2\right) v_i\left(\theta, \frac{z_i^{12} + \omega_i^2 + \beta^2\omega_i^3}{1+\beta^2}\right), \quad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1.$$

Assumption 2 (Θ^2) In this domain we drop the assumption of constant aggregate endowment over time, but we assume that agents have identical CES preferences. That is, for every $\theta \in \Theta^2$ there is a triplet (β^1, β^2, ρ) such that every i's preference ordering over consumptions is represented in the form:

$$U_i(\theta, c_i^1, c_i^2, c_i^3) = \frac{(c_i^1)^{1-\rho}}{1-\rho} + \beta^1 \frac{(c_i^2)^{1-\rho}}{1-\rho} + \beta^1\beta^2 \frac{(c_i^3)^{1-\rho}}{1-\rho}, \qquad \text{with } \rho > 0.$$

When agents have identical CES preferences, we obtain that the TP2 competitive equilibrium spot price, net trade allocations and consumption allocations prescribed for every $\theta \in \Theta^2$ are:

$$\begin{aligned}
q^2\left[\theta|z^1\right] &= \beta^2 \left(\frac{\omega^2}{\omega^3}\right)^{\rho} \\
f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{z_i^{12} + \omega_i^2 - \omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1 + \frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}} \\
f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{\omega^3}{\omega^2} \cdot \frac{z_i^{12} + \omega_i^2 - \omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1 + \beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\
c_i^{*2}\left[\theta|z^1\right] &= \frac{z_i^{12} + \omega_i^2 + \beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^{\rho}}{1 + \beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\
c_i^{*3}\left[\theta|z^1\right] &= \frac{\omega^3}{\omega^2} \cdot c_i^{*2}\left[\theta|z^1\right], \qquad\qquad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1.
\end{aligned}$$

Next, let us define a TP1 competitive equilibrium when Condition 1 is satisfied.

Definition 11 For every economy θ satisfying Condition 1, a TP1 net trade allocation $\hat{f}^1[\theta] \in \bar{Z}^1$ constitutes a *backward TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation $\hat{f}^1[\theta]$ solves the following problem:

$$\max_{z^1 \in \bar{Z}^1} \; U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + f_i^{22}\left[\theta|z_i^{12}\right], \omega_i^3 + f_i^{22}\left[\theta|z_i^{12}\right]), \quad \text{subject to } z_i^{11} + q^1[\theta]z_i^{12} \leq 0.$$

Using this definition, we obtain that the competitive equilibrium spot prices prescribed for every economy $\theta \in \Theta^1$ are:

$$q^1[\theta] = \beta^1 \text{ and } q^2\left[\theta|\hat{f}^1[\theta]\right] = \beta^2,$$

and so the competitive net trade allocations and the equilibrium consumption allocations are for every agent $i \in \mathcal{I}$ as follows:

$$\begin{aligned}
\hat{f}_i^{11}[\theta] &= -\frac{\beta^1}{1+\beta^1+\beta^1\beta^2} \cdot \left(\omega_i^1\left(1+\beta^2\right) - \omega_i^2 - \beta^2\omega_i^3\right) \\
\hat{f}_i^{12}[\theta] &= \frac{1}{1+\beta^1+\beta^1\beta^2} \cdot \left(\omega_i^1\left(1+\beta^2\right) - \omega_i^2 - \beta^2\omega_i^3\right) \\
f_i^{22}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= -\frac{\beta^2}{1+\beta^2} \cdot \left(\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\right) \\
f_i^{23}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= \frac{1}{1+\beta^2} \cdot \left(\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\right) \\
c_i^{*1}[\theta] &= c_i^{*2}\left[\theta|\hat{f}_i^{12}[\theta]\right] = c_i^{*3}\left[\theta|\hat{f}_i^{12}[\theta]\right] = \frac{\omega_i^1 + \beta^1\omega_i^2 + \beta^1\beta^2\omega_i^3}{1+\beta^1+\beta^1\beta^2}.
\end{aligned}$$

For economies in Θ^2, we obtain that the equilibrium spot prices prescribed for every $\theta \in \Theta^2$ are:

$$q^1[\theta] = \beta^1\left(\frac{\omega^1}{\omega^2}\right)^\rho \text{ and } q^2\left[\theta|\hat{f}^1[\theta]\right] = \beta^2\left(\frac{\omega^2}{\omega^3}\right)^\rho.$$

Thus, the competitive net trade allocations are:

$$\begin{aligned}
\hat{f}_i^{11}[\theta] &= -\frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right) - \left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2 + \beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^\rho\right)}{1+\frac{1}{\beta^1}\left(\frac{\omega^1}{\omega^2}\right)^{1-\rho} + \beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\
\hat{f}_i^{12}[\theta] &= \frac{\omega^2}{\omega^1} \cdot \frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right) - \left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2 + \beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^\rho\right)}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho} + \beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}} \\
f_i^{22}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= -\frac{\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}} \\
f_i^{23}\left[\theta|\hat{f}_i^{12}[\theta]\right] &= \frac{\omega^3}{\omega^2} \cdot \frac{\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}},
\end{aligned}$$

while the corresponding equilibrium consumption allocations are:

$$\begin{aligned}
c_i^{*1}[\theta] &= \frac{\omega_i^1 + \beta^1\omega_i^2\left(\frac{\omega^1}{\omega^2}\right)^\rho + \beta^1\beta^2\omega_i^3\left(\frac{\omega^1}{\omega^3}\right)^\rho}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho} + \beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}} \\
c_i^{*2}\left[\theta|z^{*1}[\theta]\right] &= \frac{\omega^2}{\omega^1} \cdot c_i^{*1}[\theta] \\
c_i^{*3}\left[\theta|z^{*1}[\theta]\right] &= \frac{\omega^3}{\omega^1} \cdot c_i^{*1}[\theta].
\end{aligned}$$

The *backward competitive solution* of an economy θ is a SCF $\bar{f} = \left(\bar{f}^1[\cdot], \bar{f}^2[\cdot|\cdot]\right)$ associating the period-1 function $\bar{f}^1[\theta]$ with the backward TP1 competitive net trade allocation $\hat{f}^1[\theta]$, that is, $\bar{f}^1[\theta] = \hat{f}^1[\theta] \in \bar{Z}^1$, and the period-2 function $\bar{f}^2[\theta|\cdot]$ with the TP2 competitive net trade allocation for any TP1 net trade allocation in the set $\bar{Z}^1$, that is, $\bar{f}^2[\theta|z^1] = f^2[\theta|z^1]$ for every $z^1 \in \bar{Z}^1$. Thanks to Condition 1, we can now state and prove the following permissive results.

Claim 2 Assume that $I \geq 3$. Suppose that the quantity ω_i^t is strictly positive for every agent i and every consumption period $t = 1, 2, 3$. Then, the backward competitive solution $\bar{f}$ is one-step-ahead implementable if it is defined either over Θ^1 or over Θ^2.

Proof. Let the premises hold. To show that $\bar{f}$ is one-step-ahead implementable when it is defined either over Θ^1 or over Θ^2, we need to show that this solution satisfies the folding condition and one-step-ahead Maskin monotonicity. Moreover, we need also to show this solution satisfies one-step-ahead unanimity and one-step-ahead weak no veto-power.

First, let us show that $\bar{f}$ satisfies the folding condition. To this end, let $Y^1 = \mathcal{Y}^{-2} = \bar{Z}^1$ and let $Y^2(z^1) = \bar{Z}^2(z^1)$ for every $z^1 \in \bar{Z}^1$. Then, the sets $Y^1 = \mathcal{Y}^{-2}$ and $Y^2(z^1)$ are not empty sets. Note that for $k = 1, 2$, it holds that $\bar{f}^1\left[\Theta^k\right] \subseteq \bar{Z}^1$ and $\bar{f}^2\left[\Theta^k|z^1\right] \subseteq \bar{Z}^2(z^1)$ for every $z^1 \in \bar{Z}^1$.

Let us define the TP2 induced ordering of agent i in state θ at $z^1 \in \bar{Z}^1$, denoted by $R_i[\theta|z^1]$, as follows:

$$
\begin{array}{rcl}
x^2 R_i\left[\theta|z^1\right] y^2 & \iff & \\
U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + x_i^{22}, \omega_i^3 + x_i^{23}) & \geq & U_i(\theta, \omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + y_i^{22}, \omega_i^3 + y_i^{23}),
\end{array}
$$

for every $x^2, y^2 \in Y^2(z^1)$. We denote by $R[\theta|z^1]$ the profile of TP2 induced orderings at $\theta|z^1$, by $\mathcal{D}[\Theta^1|z^1]$ the TP2 domain of induced orderings at $\Theta^1|z^1$ and by $\mathcal{D}[\Theta^2|z^1]$ the TP2 domain of induced orderings at $\Theta^2|z^1$. For every $k = 1, 2$, let us define the TP2 function $\varphi^2 : \mathcal{D}\left[\Theta^k|z^1\right] \rightarrow Y^2(z^1)$ as follows:

$$
\varphi^2\left(R\left[\theta|z^1\right]\right) = \bar{f}^2\left[\theta|z^1\right], \qquad \forall \theta \in \Theta^k.
$$

The TP1 induced ordering of agent i in state θ, denoted by $R_i\left[\theta|\bar{f}^2\right]$, is defined as in (11). Let us denote by $R\left[\theta|\bar{f}^2\right]$ the profile of TP1 induced orderings at $\theta|\bar{f}^2$, by $\mathcal{D}\left[\Theta^1|\bar{f}^2\right]$ the TP1 domain of induced orderings at $\Theta^1|\bar{f}^2$, and by $\mathcal{D}\left[\Theta^2|\bar{f}^2\right]$ the TP1 domain of induced orderings at $\Theta^2|\bar{f}^2$. For every $k = 1, 2$, let us define the TP1 function $\varphi^1 : \mathcal{D}\left[\Theta^k|\bar{f}^2\right] \rightarrow Y^1$ as follows:

$$
\varphi^1\left(R\left[\theta|\bar{f}^2\right]\right) = \bar{f}^1[\theta], \qquad \forall \theta \in \Theta^k.
$$

By the above definitions and by the fact that competitive equilibrium exists in each TP, one can check that the backward competitive solution $\bar{f}$ satisfies the folding condition.

To see that $\bar{f}$ also satisfies one-step-ahead Maskin monotonicity, it suffices to observe that in each TP the competitive net trade allocation is unique and always an interior allocation, that the TP1 competitive solution φ^1 on $\mathcal{D}\left[\Theta^k|\bar{f}^2\right]$ is Maskin monotonic for $k = 1, 2$, and that the TP2 competitive solution φ^2 on $\mathcal{D}\left[\Theta^k|z^1\right]$ is also Maskin monotonic for $k = 1, 2$.

Finally, to see that the backward competitive solution $\bar{f}$ satisfies one-step-ahead unanimity and one-step-ahead weak no veto-power it suffices to observe that they are vacuously satisfied since agents' induced orderings are strictly monotonic in consumption. ■

4.2 Period-by-period implementability of the Condorcet winner

In this section, we consider a bi-dimensional policy space where an *odd* number of agents vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This stage-by-stage resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). We are interested in one-step-ahead implementing the simple majority solution, which selects the Condorcet winner in each voting stage.

A policy choice is an ordered pair $(x^1, x^2) \in X^1 \times X^2$, where the policy space of dimension $d = 1, 2$ is an open interval.[7] Each voter i is described by a one-dimensional type θ_i. The type space is the open interval $\left(\underline{\eta}, \bar{\eta}\right)$.

Definition 12 The voter i's utility function $U : \left(\underline{\eta}, \bar{\eta}\right) \times X^1 \times X^2 \to \mathbb{R}$ is a twice-continuously differentiable satisfying:
(a) *Strict concavity*, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial^2 x^1} < 0 \quad \text{and} \quad \frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial^2 x^2} < 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2.$$

(b) *induced single-crossing* property, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial \theta_i \partial x^1} > 0 \quad \text{and} \quad \frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial \theta_i \partial x^2} > 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2 \text{ and } \theta_i \in \left(\underline{\eta}, \bar{\eta}\right).$$

(c) *Strategic complementarity*, that is:

$$\frac{\partial^2 U(\theta_i, x^1, x^2)}{\partial x^1 \partial x^2} \geq 0, \quad \text{for every } \left(x^1, x^2\right) \in X^1 \times X^2.$$

The induced single-crossing property simply requires that the induced utility of both dimensions is increasing in the type of voter. This property can also be found in De Donder et al. (2012).

We now introduce the definition of a Condorcet winner for an arbitrary policy space P:

Definition 13 Suppose that agents in $\mathcal{I}$ votes over the set of policies P. We say that $p \in P$ is a majority voting outcome, also known as a *Condorcet winner* (CW), if there does not exist any other distinct outcome $p' \in P$ that is strictly preferred by more than half of voters to the outcome p.

For any integer $k \geq 2$, the set of states Θ takes the structure of the Cartesian product of allowable independent types for voters, that is, $\Theta \equiv \left(\underline{\eta}, \bar{\eta}\right)^{2k-1}$, with θ as typical element.

[7] The choice of a bi-dimensional policy space is motivated by convenience.

It simplifies the argument, and causes no loss of generality, to assume that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{2k-1}$. Therefore, the type θ_k is the median type, denoted by θ_{med}, at state θ.

At the state θ, each voter is assumed to have an ordering preference relation $R_i(\theta)$ over the policy space $X^1 \times X^2$ which is represented by $U(\theta_i, \cdot, \cdot)$.

Solving by backward induction when the state θ is the prevailing state, if $x^1 \in X^1$ is the outcome of the first majority voting, then the stage-2 induced ordering of voter i on X^2 in state θ at x^1 is denoted by $R_i[\theta|x^1]$ and is represented by $U(\theta_i, x^1, \cdot)$.

The profile of the stage-2 induced orderings in state θ at x^1 is denoted by $R[\theta|x^1]$. Let $\mathcal{D}[\Theta|x^1]$ be the stage-2 domain of induced ordering preferences induced by the set Θ as well as by the outcome x^1; that is:

$$\mathcal{D}\left[\Theta|x^1\right] \equiv \left\{R\left[\theta|x^1\right]|\theta \in \Theta\right\}, \quad \text{for every } x^1 \in X^1. \tag{12}$$

If $x^1 \in X^1$ is the outcome of the first majority voting, then the stage-2 majority voting function $f^2 : \mathcal{D}[\Theta|x^1] \rightarrow X^2$ is defined as follows:

$$f^2\left[\theta|x^1\right] = CW\left(R\left[\theta|x^1\right]\right),$$

where $CW(R[\theta|x^1])$ denotes the Condorcet winner under the profile $R[\theta|x^1]$. It will be shown below that this outcome is the most-preferred outcome of the median type.

Let us suppose that the stage-2 majority voting function is well-defined for every outcome $x^1 \in X^1$. Then, in stage-1, the utility of a voter i at state θ for the outcome $z^1 \in X^1$ is:

$$U\left(\theta_i, z^1, f^2\left[\theta|z^1\right]\right).$$

Then, the stage-1 induced ordering of voter i on X^1 in state θ at the majority voting function $f^2[\theta|\cdot]$, denoted by $R_i[\theta|f^2]$, is given by:

$$y^1 R_i\left[\theta|f^2\right] z^1 \iff \left(y^1, f^2\left[\theta|y^1\right]\right) R_i(\theta) \left(z^1, f^2\left[\theta|z^1\right]\right), \quad \text{for every } y^1, z^1 \in X^1.$$

As usual, the profile of the stage-1 induced orderings in state θ at the majority voting function $f^2[\theta|\cdot]$ is denoted by $R[\theta|f^2]$. Let $\mathcal{D}[\Theta|f^2]$ be the stage-1 domain of induced ordering preferences induced by the set Θ as well as by the majority voting function f^2; that is:

$$\mathcal{D}\left[\Theta|f^2\right] \equiv \left\{R\left[\theta|f^2\right]|\theta \in \Theta\right\}. \tag{13}$$

Thus, the stage-1 majority voting function $f^1 : \mathcal{D}[\Theta|f^2] \rightarrow X^1$ is defined as follows:

$$f^1[\theta] = CW\left(R\left[\theta|f^2\right]\right), \quad \text{for every } \theta \in \Theta,$$

where $CW(R[\theta|f^2])$ denotes the Condorcet winner under the profile $R[\theta|f^2]$.

Definition 14 The SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ on Θ is the *majority voting* solution if for every $\theta \in \Theta$:

$$f^1[\theta] = CW\left(R\left[\theta|f^2\right]\right) \text{ and } f^2\left[\theta|x^1\right] = CW\left(R\left[\theta|x^1\right]\right) \text{ for every } x^1 \in X^1.$$

The following lemma shows that the majority voting solution is a single-valued function.

The intuition behind it is similar to that of Proposition 4 of De Donder et al. (2012) for the case where there is a continuum of voters. Firstly, the assumption of strict concavity assures the existence and unicity of the Condorcet winner in the second voting stage. This assumption, combined with the assumption of strategic complementarity and with the induced single-crossing property, assures that the stage-1 induced ordering of voter i on X^1 in state θ at the majority voting function $f^2[\theta|\cdot]$ is single-crossing. This guarantees the existence and unicity of the Condorcet winner in the first voting stage.

Lemma 1 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $\Theta \times X^1 \times X^2$ meets the requirements of Definition 12 and depends only on her own type. Then, the majority voting SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ over Θ is a single-valued function on each policy dimension.

Proof. See Appendix. ■

Thanks to the above lemma, we can now state and prove the main result of this section.

Claim 3 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $\Theta \times X^1 \times X^2$ meets the requirements of Definition 12 and depends only on her own type. Then, the majority voting solution is one-step-ahead implementable.

Proof. Let the premises hold. By Theorem 3, it suffices to show that the majority voting solution satisfies the folding condition and one-step-ahead Maskin monotonicity and, moreover, it satisfies one-step-ahead unanimity and one-step-ahead weak no veto-power.

Thus, $\mathcal{T} = \{1, 2\}$. Let $Y^1 = X^1$ and $Y^2(x^1) = X^2$ for every $x^1 \in X^1$. Define $\mathcal{D}[\Theta|x^1]$ as in (12) and define $\mathcal{D}[\Theta|f^2]$ as in (13).

For every $x^1 \in X^1$, define the second stage function $\varphi^2 : \mathcal{D}[\Theta|x^1] \to X^2$ by $\varphi^2(R[\theta|x^1]) = CW(R[\theta|x^1])$ for every $R[\theta|x^1] \in \mathcal{D}[\Theta|x^1]$. Moreover, define the first stage function $\varphi^1 : \mathcal{D}[\Theta|f^2] \to X^1$ by $\varphi^1(R[\theta|f^2]) = CW(R[\theta|f^2])$ for every $R[\theta|f^2] \in \mathcal{D}[\Theta|f^2]$. These functions are single-valued by Lemma 1. This shows that the majority voting solution satisfies the folding condition.

By definitions of the preceding paragraph and by the fact that in each period agents have single crossing preferences, one can see that the majority voting solution satisfies one-step-ahead Maskin monotonicity. Since unanimity and weak no veto-power are satisfied, we conclude that the majority voting solution on Θ is one-step-ahead implementable. ■

5. Conclusion

Summary. We have identified two necessary conditions for one-step-ahead implementability, *the folding condition* and *one-step-ahead Maskin monotonicity.* The first condition states that a one-step-ahead implementable SCF can be decomposed into a sequence of social choice functions, each of which is defined only over induced preferences induced over outcomes at hand. Each induced preference is constructed in the manner of backward-induction. This means that a period-t induced preference over the current component set depends on past decisions as well as on the socially optimal path that the dynamic process will bring about in the future. The second condition states that every such social function needs to satisfy a remarkably strong invariance condition for Nash implementation, now widely referred to as Maskin monotonicity (Maskin, 1999). We have also shown that under two auxiliary conditions the two necessary conditions are sufficient, as well.

We have applied our analysis to two prominent dynamic problems, voting over time and sequential trading. In the voting application, we have shown that on the domain satisfying the single-crossing property the simple majority solution, which selects the Condorcet winner in each voting stage (after every history), is one-step-ahead implementable.

In a borrowing-lending model with no liquidity constraints, in which agents trade in spot markets and transfer wealth between any two periods by borrowing and lending, we have noted that intertemporal pecuniary externalities arise because trades in the current period change the spot price of the next period, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every agent's induced preference ordering concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other agents. In this set-up, we have shown that, under such pecuniary externalities, the standard dynamic competitive equilibrium solution is not one-step-ahead implementable. However, we have also identified preference domains – which involve no pecuniary externalities – for which the no-commitment version of the dynamic competitive equilibrium solution is definable and one-step-ahead implementable. It remains an open question how we should deal with intertemporal pecuniary externalities. We hope that this and other topics related to this paper will be investigated in future research.

In this paper, we have considered one-step-ahead implementation in SPE. One may consider that agents follow an alternative equilibrium concept, which is known to be less restrictive. This does not ease the restrictiveness of one-step-ahead implementability, however, since it does not change the restrictiveness of the folding condition.

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Appendix

Proof of Theorem 1

Proof of Theorem 1. Let the premises hold. Thus, there exists a dynamic mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ that one-step-ahead implements the SCF f. Therefore, for every $\bar{\theta} \in \Theta$,

$$f^1\left[\bar{\theta}\right] = g^1\left(s^{\bar{\theta}}\left(h^1\right)\right) \text{ and}$$

$$f^t\left[\bar{\theta}|g^{-t}\left(h^t\right)\right]=g^t\left(s^{\bar{\theta}}\left(h^t\right)\right) \text{ for every } h^t\in H^t \text{ with } t\neq 1$$

if and only if $s^{\bar{\theta}}\in SPE\left(\Gamma,\bar{\theta}\right)$. Fix any $s^{\bar{\theta}}\in SPE\left(\Gamma,\bar{\theta}\right)$. Then, $s^{\bar{\theta}}|h^t$ is a Nash equilibrium of $\left(\Gamma\left(h^t\right),\bar{\theta}\right)$ for every history $h^t\in H$. Moreover, by one-step-ahead implementability of f, it also follows that:

$$f^{+t}\left[\bar{\theta}|g^{-t}\left(h\right)\right]=g^{+t}\left(s^{\bar{\theta}}|h\right),\quad \text{for every } h\in H^t \text{ with } 2\leq t\leq T. \tag{14}$$

Fix any period $t\neq 1$. Let us define the set Y^1, the set $\mathcal{Y}^{-t}$ and the set $Y^t\left(g^{-t}\left(h\right)\right)$ as follows:

$$Y^1\equiv\left\{g^1\left(a\left(h^1\right)\right)\in X^t|\text{for some } a\left(h^1\right)\in A\left(h^1\right)\right\}, \tag{15}$$

$$\mathcal{Y}^{-t}\equiv\left\{g^{-t}\left(h\right)\in\mathcal{X}^{-t}|\text{for some } h\in H^t\right\}, \tag{16}$$

and for every $g^{-t}\left(h\right)\in\mathcal{Y}^{-t}$:

$$Y^t\left(g^{-t}\left(h\right)\right)\equiv\left\{g^t\left(a\left(h\right)\right)\in X^t\left(g^{-t}\left(h\right)\right)|a\left(h\right)\in A\left(h\right)\ \text{ for some } h\in H^t\right\}. \tag{17}$$

By their definitions as well as by the assumption that the dynamic mechanism Γ implements in SPE the SCF f, one can check that $f^t\left[\Theta|g^{-t}\left(h\right)\right]\subseteq Y^t\left(g^{-t}\left(h\right)\right)$ and that $f^1\left[\Theta\right]\subseteq Y^1$.

Moreover, given that Γ is a dynamic mechanism, one can also check that for every period $t\neq 1$:

$$g^{-t}\left(h^t\right)\in\mathcal{Y}^{-t}\iff g^1\left(a^1\right)\in Y^1 \text{ and } g^\tau\left(a^\tau\right)\in Y^\tau\left(g^{-\tau}\left(a^1,\cdots,a^{\tau-1}\right)\right)$$

for every τ such that $2\leq\tau\leq t-1$, for every $h^t\equiv\left(a^1,\cdots,a^{t-1}\right)\in H^t$.

For every $y^{-T}\in\mathcal{Y}^{-T}$, the period-$T$ preference domain $\mathcal{D}\left[\Theta|y^{-T}\right]$ is nonempty, and this follows from its definition in (2) and from the fact that $Y^T\left(y^{-T}\right)$ is not empty. Let the period-T function

$$\varphi^T:\mathcal{D}\left[\Theta|g^{-T}\left(h\right)\right]\to Y^T\left(g^{-T}\left(h\right)\right)$$

be defined by:

$$\varphi^T\left(R\left[\theta|g^{-T}\left(h\right)\right]\right)=g^T(s^\theta\left(h\right)),\quad \text{for every history } h\in H^T \text{ and state } \theta\in\Theta, \tag{18}$$

where $s^\theta\in SPE\left(\Gamma,\theta\right)$.

Fix any period $t\neq 1,T$ and any t-head outcome path $y^{-t}\equiv g^{-t}\left(h\right)\in\mathcal{Y}^{-t}$ for some $h\in H^t$. Since the set $Y^t\left(g^{-t}\left(h\right)\right)$ is not empty and since Γ one-step-ahead implements f, one can see that the period-t domain of induced orderings $\mathcal{D}\left[\Theta|y^{-t},f^{+(t+1)}\right]$ as defined in (5) is not empty. Similarly, one can see that period-1 domain of induced orderings $\mathcal{D}\left[\left(\Theta|f^{+2}\right)\right]$ as defined in (8) is not empty.

For every $t\neq 1,T$, let the period-t function

$$\varphi^t:\mathcal{D}\left[\Theta|g^{-t}\left(h\right),f^{+(t+1)}\right]\to Y^t\left(g^{-t}\left(h\right)\right)$$

be defined by:

$$\varphi^t\left(R\left[\theta|g^{-t}\left(h\right),f^{+(t+1)}\right]\right)=g^t(s^\theta\left(h\right)) \text{ for every } h\in H^t \text{ and every } \theta\in\Theta. \tag{19}$$

Let the period-1 function

$$\varphi^1 : \mathcal{D}\left[\Theta|f^{+2}\right] \to Y^1$$

be defined by:

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) = g^1(s^\theta\left(h^1\right)), \quad \text{for every } \theta \in \Theta. \tag{20}$$

To complete the proof, we need to show that the period-t function φ^t is a function for every $t \in \mathcal{T}$. The following claim establishes it for the case where $t \neq 1, T$. The same arguments, suitably modified, can be used to show that φ^1 and φ^T are functions.

Claim 4 If the SCF f over Θ is one-step-ahead implementable and $R\left[\theta|y^{-t}, f^{+(t+1)}\right] = R\left[\theta'|y^{-t}, f^{+(t+1)}\right]$ for some $y^{-t} \in \mathcal{Y}^{-t}$ with $t \neq 1$ and some $\theta, \theta' \in \Theta$, then $f^t\left[\theta|y^{-t}\right] = f^t\left[\theta'|y^{-t}\right]$.

Proof. Suppose that $y^{-t} = g^{-t}(h)$ for some $h \in H^t$ and that $R\left[\theta|y^{-t}, f^{+(t+1)}\right] = R\left[\theta'|y^{-t}, f^{+(t+1)}\right]$ for some $\theta, \theta' \in \Theta$.

Since $s^\theta \in SPE(\Gamma, \theta)$ and since, moreover, $R\left[\theta|y^{-t}, f^{+(t+1)}\right] = R\left[\theta'|y^{-t}, f^{+(t+1)}\right]$, we have that:

$$s^\theta(h) \in NE\left(\Gamma(h), R\left[\theta|y^{-t}, f^{+(t+1)}\right]\right) \cap NE\left(\Gamma(h), R\left[\theta'|y^{-t}, f^{+(t+1)}\right]\right),$$

and so, for every $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$, it holds that:

$$s^\theta(h)\, R_i\left[\theta'|y^{-t}, f^{+(t+1)}\right]\left(a_i(h), s^\theta_{-i}(h)\right).$$

From the definition of $R_i\left[\theta'|y^{-t}, f^{+(t+1)}\right]$ and from (14), it follows that for every $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$ it holds that:

$$\begin{aligned}&\left(g^{-t}(h), g^t\left(s^\theta(h)\right), g^{+(t+1)}\left(s^{\theta'}|\left(h, s^\theta(h)\right)\right)\right) R_i(\theta') \\ &\left(g^{-t}(h), g^t\left(a_i(h), s^\theta_{-i}(h)\right), g^{+(t+1)}\left(s^{\theta'}|\left(h, \left(a_i(h), s^\theta_{-i}(h)\right)\right)\right)\right).\end{aligned} \tag{21}$$

Let s_i denote agent i's strategy according to which this i plays $s_i(h') = s_i^{\theta'}(h')$ for every history $h' \neq h$ and according to which this i plays $s_i^t = s_i^\theta(h)$ after the history h. Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every history $h' \neq h$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile s is a SPE strategy profile of (Γ, θ'), we need to show that $s|h$ is a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the action profile $s(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, it follows that (21) holds for every $i \in I$ and every $a_i(h) \in A_i(h)$. Thus, no agent i can gain by deviating from the action $s_i(h)$ and thereafter conforming to s_i. Since the one deviation property (see, e.g., Osborne and Rubinstein, 1994; Lemma 98.2) holds for a finite-horizon multi-period game with observed actions and simultaneous moves, it follows that the strategy profile $s|h \in SPE(\Gamma(h), \theta')$, and so $s|h \in NE(\Gamma, \theta')$. Therefore, we have that $s \in SPE(\Gamma, \theta')$. Since the dynamic mechanism Γ implements the SCF f in SPE and $g\left(s^\theta(h)\right) = g(s(h))$ we have that $f^t\left[\theta'|g^{-t}(h)\right] = f^t\left[\theta|g^{-t}(h)\right]$. ■

The statement follows by the above arguments. ■

Proof of Theorem 2

Proof of Theorem 2. Let the premises hold. Thus, there exists a dynamic mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ that one-step-ahead implements the SCF f. Therefore, for every $\bar{\theta} \in \Theta$,

$$f^1\left[\bar{\theta}\right] = g^1\left(s^{\bar{\theta}}\left(h^1\right)\right) \text{ and}$$

$$f^t\left[\bar{\theta}|g^{-t}\left(h^t\right)\right] = g^t\left(s^{\bar{\theta}}\left(h^t\right)\right) \text{ for every } h^t \in H^t \text{ with } t \neq 1$$

if and only if $s^{\bar{\theta}} \in SPE\left(\Gamma, \bar{\theta}\right)$. Consider any state $\bar{\theta}$. Fix any $s^{\bar{\theta}} \in SPE\left(\Gamma, \bar{\theta}\right)$. Then, $s^{\bar{\theta}}|h^t$ is a Nash equilibrium of $\left(\Gamma\left(h^t\right), \bar{\theta}\right)$ for every history $h^t \in H$. Moreover, by one-step-ahead implementability of f, it also follows that:

$$f^{+t}\left[\bar{\theta}|g^{-t}(h)\right] = g^{+t}\left(s^{\bar{\theta}}|h\right), \quad \text{for every } h \in H^t \text{ with } 2 \leq t \leq T.$$

Since the SCF f satisfies the folding condition, define the set Y^1, the set $\mathcal{Y}^{-t}$ and the set $Y^t\left(g^{-t}\left(h^t\right)\right)$ as in (15), (16) and (17) of the proof of Theorem 1, respectively.

Fix any $g^{-T}(h) \in \mathcal{Y}^{-T}$ with $h \in H^T$ and suppose that for every $i \in \mathcal{I}$ and every $a(h) \in A(h)$, it holds that:

$$\begin{aligned} &\varphi^T\left(R\left[\theta|g^{-T}(h)\right]\right) R_i\left[\theta|g^{-T}(h)\right] g^T(a(h)) \implies \\ &\qquad \varphi^T\left(R\left[\theta|g^{-T}(h)\right]\right) R_i\left[\theta'|g^{-T}(h)\right] g^T(a(h)), \end{aligned} \tag{22}$$

for some $R\left[\theta|g^{-T}(h)\right]$ and $R\left[\theta'|g^{-T}(h)\right]$ in $\mathcal{D}\left[\Theta|g^{-T}(h)\right]$.

Since the dynamic mechanism Γ implements the SCF f in SPE, we have that:

$$\varphi^T\left(R\left[\theta|g^{-T}(h)\right]\right) = g^T(s^\theta(h)) = f^T\left[\theta|g^{-T}(h)\right],$$

and that action profile $s^\theta(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta|g^{-T}(h)\right]\right)$.

From the definitions of $R_i\left[\theta|g^{-T}(h)\right]$ and $R_i\left[\theta'|g^{-T}(h)\right]$ given in (1), we have that:

$$\begin{aligned} &g^T(s^\theta(h))R_i\left[\theta|g^{-T}(h)\right] g^T(a(h)) \iff \\ &\qquad \left(g^{-T}(h), g^T(s^\theta(h))\right) R_i(\theta) \left(g^{-T}(h), g^T(a(h))\right), \end{aligned} \tag{23}$$

and that:

$$\begin{aligned} &g^T(s^\theta(h))R_i\left[\theta'|g^{-T}(h)\right] g^T(a(h)) \iff \\ &\qquad \left(g^{-T}(h), g^T(s^\theta(h))\right) R_i(\theta') \left(g^{-T}(h), g^T(a(h))\right). \end{aligned} \tag{24}$$

If there exist $i \in \mathcal{I}$ and $a_i(h) \in A_i(h)$ such that:

$$g^T\left(a_i(h), s^\theta_{-i}(h)\right) P_i\left[\theta'|g^{-T}(h)\right] g^T(s^\theta(h)),$$

it follows from (22)-(24) that:

$$g^T\left(a_i(h), s^\theta_{-i}(\theta)(h)\right) P_i\left[\theta|g^{-T}(h)\right] g^T(s^\theta(h)),$$

which contradicts the fact that the action profile $s^{\theta}(h)$ is a Nash equilibrium of $\left(\Gamma\left(h^{T}\right), R\left[\theta|g^{-T}(h)\right]\right)$. Thus, this action profile $s^{\theta}(h)$ is also a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-T}(h)\right]\right)$. Also, note that this profile $s^{\theta}(h)$ is also a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the period-T SCF f^{T} is a function and since the action profile $s^{\theta}(h)$ is a Nash equilibrium of $(\Gamma(h), \theta')$, it needs to be the case that $g^{T}\left(s^{\theta}(h)\right) = f^{T}\left(\theta'|g^{-T}(h)\right)$. It follows from the fact that the SCF f satisfies the folding condition that $g^{T}\left(s^{\theta}(h)\right) = \varphi^{T}\left(R\left[\theta'|g^{-T}(h)\right]\right)$, as was to be proved.

Fix any $t \neq 1, T$ and consider any $g^{-t}(h) \in \mathcal{Y}^{-t}$ with $h \in H^{t}$. Furthermore, consider any profile $R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]$ and any profile $R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]$ in $\mathcal{D}\left[\Theta|g^{-t}(h), f^{+(t+1)}\right]$. Suppose that for every $i \in \mathcal{I}$ and every $a(h) \in A(h)$:

$$\begin{aligned} &\varphi^{t}\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) R_{i}\left[\theta|g^{-t}(h), f^{+(t+1)}\right] g^{t}(a(h)) \Longrightarrow \\ &\qquad \varphi^{t}\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) R_{i}\left[\theta'|g^{-t}(h), f^{+(t+1)}\right] g^{t}(a(h)). \end{aligned} \tag{25}$$

Since the dynamic mechanism Γ implements the SCF f in SPE, we have that:

$$\varphi^{t}\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right) = f^{t}\left[\theta|g^{-t}(h)\right] = g^{t}\left(s^{\theta}(h)\right).$$

Moreover, from the definitions of $R_{i}\left[\theta|g^{-t}(h), f^{+(t+1)}\right]$ and $R_{i}\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]$ given in (4) and from the fact that Γ one-step-ahead implements the SCF f, one can see that the action profile $s^{\theta}(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right)$, that:

$$\begin{aligned} &g^{t}\left(s^{\theta}(h)\right) R_{i}\left[\theta|g^{-t}(h), f^{+(t+1)}\right] g^{t}(a(h)) \Longleftrightarrow \\ &\quad \left(g^{-t}(h), g^{+t}\left(s^{\theta}|h\right)\right) R_{i}(\theta)\left(g^{-t}(h), g^{t}(a(h)), g^{+(t+1)}\left(s^{\theta}|(h, a(h))\right)\right), \end{aligned} \tag{26}$$

and that:

$$\begin{aligned} &g^{t}\left(s^{\theta}(h)\right) R_{i}\left[\theta'|g^{-t}(h), f^{+(t+1)}\right] g^{t}(a(h)) \Longleftrightarrow \\ &\left(g^{-t}(h), g^{t}\left(s^{\theta}(h)\right), g^{+(t+1)}\left(s^{\theta'}|\left(h, s^{\theta}(h)\right)\right)\right) R_{i}(\theta')\left(g^{-t}(h), g^{t}(a(h)), g^{+(t+1)}\left(s^{\theta'}|(h, a(h))\right)\right). \end{aligned} \tag{27}$$

If there exist $i \in \mathcal{I}$ and $a_{i}(h) \in A_{i}(h)$ such that:

$$g^{t}\left(a_{i}(h), s_{-i}^{\theta}(h)\right) P_{i}\left[\theta'|g^{-t}(h), g^{+(t+1)}\left(s^{\theta'}|(h, \cdot)\right)\right] g^{t}(s^{\theta}(h)),$$

it follows from (25)-(27) that:

$$g^{t}\left(a_{i}(h), s_{-i}^{\theta}(h)\right) P_{i}\left[\theta|g^{-t}(h), g^{+(t+1)}\left(s^{\theta}|(h, \cdot)\right)\right] g^{t}(s^{\theta}(h)),$$

which contradicts the fact that $s^{\theta}(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right)$. Thus, the action profile $s^{\theta}(h)$ is also a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$.

Let s_{i} denote agent i's strategy according to which this i plays $s_{i}(h') = s_{i}^{\theta'}(h')$ for every history $h' \neq h$ and according to which this i plays $s_{i}^{t} = s_{i}^{\theta}(h)$ after the history h. Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every history $h' \neq h$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile s is a SPE strategy profile of (Γ, θ'), we need to show that $s|h$ is a Nash equilibrium of $(\Gamma(h), \theta')$.

Since the action profile $s(h)$ is a Nash equilibrium of $\left(\Gamma(h), R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, it follows from (27) that for every $i \in I$ and every $a_i(h) \in A_i(h)$:

$$\left(g^{-t}(h), g^{+t}(s)\right) R_i(\theta') \left(g^{-t}(h), g^t(a_i(h), s_{-i}(h)), g^{+(t+1)}(s|(h, (a_i(h), s_{-i}(h))))\right).$$

Thus, no agent i can gain by deviating from the action profile $s(h)$ and thereafter conforming to s_i, and so the strategy profile $s(h)$ is a NE of $(\Gamma(h), \theta')$. It follows that $s \in SPE(\Gamma, \theta')$.

Since the dynamic mechanism Γ implements the SCF f in SPE and since, moreover, the strategy profile $s \in SPE(\Gamma, \theta')$, it follows that $f^t[\theta'|g^{-t}(h)] = g^t(s(h))$. Since f satisfies the folding condition, we have $g^t(s(h)) = \varphi^t\left(R\left[\theta'|g^{-t}(h), f^{+(t+1)}\right]\right)$, as was to be shown.

Consider some $R[\theta|f^{+2}]$ and some $R[\theta'|f^{+2}]$ in $\mathcal{D}[\Theta|f^{+2}]$. Suppose that for every $i \in \mathcal{I}$ and every $a(h^1) \in A(h^1)$:

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) R_i\left[\theta|f^{+2}\right] g^1\left(a\left(h^1\right)\right) \implies \varphi^1\left(R\left[\theta|f^{+2}\right]\right) R_i\left[\theta'|f^{+2}\right] g^1\left(a\left(h^1\right)\right). \tag{28}$$

Since f satisfies the folding condition, we have that

$$\varphi^1\left(R\left[\theta|f^{+2}\right]\right) = f^1[\theta] = g^1\left(s^\theta\left(h^1\right)\right).$$

Moreover, it also follows from the definitions of $R_i[\theta|f^{+2}]$ and $R_i[\theta'|f^{+2}]$ given in (7) and from the fact that Γ one-step-ahead implements the SCF f that the action profile $s^\theta(h^1)$ is a Nash equilibrium of $(\Gamma(h^1), R[\theta|f^{+2}])$, that:

$$\begin{aligned}&\varphi^1(R[\theta|f^{+2}]) R_i[\theta|f^{+2}] g^1(a(h^1)) \iff \\ &\qquad \left(g^1\left(s^\theta\left(h^1\right)\right), g^{+2}\left(s^\theta|s^\theta\left(h^1\right)\right)\right) R_i(\theta)\left(g^1\left(a\left(h^1\right)\right), g^{+2}\left(s^\theta|a\left(h^1\right)\right)\right),\end{aligned} \tag{29}$$

and that:

$$\begin{aligned}&\varphi^1(R[\theta|f^{+2}]) R_i[\theta'|f^{+2}] g^1(a(h^1)) \iff \\ &\qquad \left(g^1\left(s^\theta\left(h^1\right)\right), g^{+2}\left(s^{\theta'}|s^\theta\left(h^1\right)\right)\right) R_i(\theta')\left(g^1\left(a\left(h^1\right)\right), g^{+2}\left(s^{\theta'}|a\left(h^1\right)\right)\right).\end{aligned} \tag{30}$$

Suppose that

$$g^1\left(a_i\left(h^1\right), s^\theta_{-i}\left(h^1\right)\right) P_i\left(\theta'|f^{+2}\right) g^1\left(s^\theta\left(h^1\right)\right)$$

for some $i \in I$ and some $a_i(h^1) \in A_i(h^1)$. Thus, it follows from (28)-(30) that:

$$\begin{aligned}&g^1\left(a_i\left(h^1\right), s^\theta_{-i}\left(h^1\right)\right) P_i\left(\theta|f^{+2}\right) g^1\left(s^\theta\left(h^1\right)\right) \iff \\ &\quad \left(g^1\left(a_i\left(h^1\right), s^\theta_{-i}\left(h^1\right)\right), g^{+2}\left(s^\theta|\left(a_i\left(h^1\right), s^\theta_{-i}\left(h^1\right)\right)\right)\right) P_i(\theta)\left(g^1\left(s^\theta\left(h^1\right)\right), g^{+2}\left(s^\theta|s^\theta\left(h^1\right)\right)\right),\end{aligned}$$

which contradicts the fact that action profile $s^\theta(h^1)$ is a Nash equilibrium of $(\Gamma(h^1), R[\theta|f^{+2}])$. Therefore, the profile $s^\theta(h^1)$ is also a Nash equilibrium of $(\Gamma(h^1), R[\theta'|f^{+2}])$.

As we did previously, let $s_i \equiv (s_i^\tau)_{\tau \geq 1}$ denote the agent i's strategy according to which this i plays $s_i^1 \equiv s_i^\theta(h^1)$ at the start of the game and thereafter she conforms to the strategy $s_i^{\theta'}$; that is, $s_i^t \equiv \left(s_i^{\theta'}\right)^t$ for every $t \geq 2$.

Note that $s|h'$ is a Nash equilibrium of $(\Gamma(h'), \theta')$ for every nontrivial history $h' \in H$ since $s^{\theta'}$ is a strategy profile in $SPE(\Gamma, \theta')$. Thus, to have that the strategy profile s is a SPE of (Γ, θ'), we need to show that s is also a Nash equilibrium of (Γ, θ').

Since the action profile $s(h^1)$ is a Nash equilibrium of $(\Gamma(h^1), R[\theta'|f^{+2}])$, it follows from

(30) that for every $i \in I$ and every $a_i(h^1) \in A_i(h^1)$:

$$(g(s))\, R_i(\theta')\left(g^1\left(a_i\left(h^1\right), s_{-i}\left(h^1\right)\right), g^{+2}\left(s|\left(a_i\left(h^1\right), s_{-i}\left(h^1\right)\right)\right)\right).$$

Thus, no agent i can gain by deviating from $s_i(h^1)$ and thereafter conforming to s_i, and so the strategy profile s is a SPE of $(\Gamma(h), \theta')$.

Since the dynamic mechanism Γ implements the SCF f in SPE, we have that $f^1[\theta'] = g^1(s(h^1))$. Since f satisfies the folding condition, we have $g^1(s(h^1)) = \varphi^1(R[\theta'|f^{+2}])$, as was to be shown. ■

Proof of Theorem 3

Proof of Theorem 3. The proof is based on the construction of a dynamic mechanism Γ, where each period-t mechanism is a canonical mechanism.

Period-1 mechanism:
Individual i's period-1 action space is defined by:

$$A_i\left(H^1\right) \equiv \mathcal{D}\left[\Theta|f^{+2}\right] \times Y^1 \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers and H^1 is the null set. Thus, a period-1 action of agent i consists of an element of the set Y^1, an element of the period-1 domain of induced preferences induced by the set Θ at the socially optimal 2-tail outcome paths f^{+2}, and a nonnegative integer. A typical period-1 action played by agent i is denoted by $a_i(h^1) \equiv \left(\left(R\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, (z)^i\right)$.

Period-1 action space of agents is the product space:

$$A\left(H^1\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(H^1\right),$$

with $a(h^1)$ as a typical period-1 action profile.

The period-1 outcome function g^1 is defined by the following three rules:

Rule 1: If $a_i(h^1) \equiv \left(R\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ for every $i \in \mathcal{I}$ and $x^1 = \varphi^1\left(R\left[\bar{\theta}|f^{+2}\right]\right)$, then $g^1(a(h)) = x^1$.

Rule 2: If $n-1$ agents play $a_j(h^1) \equiv \left(R\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ with $x^1 = \varphi^1\left(R\left[\bar{\theta}|f^{+2}\right]\right)$ but agent i plays $a_i(h^1) \equiv \left(\left(R\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, (z)^i\right) \neq a_j(h^1)$, then we can have two cases:

1. If $x^1 R_i\left[\bar{\theta}|f^{+2}\right]\left(x^1\right)^i$, then $g^1(a(h^1)) = \left(x^1\right)^i$.

2. If $\left(x^1\right)^i P_i\left[\bar{\theta}|f^{+2}\right] x^1$, then $g^1(a(h^1)) = x^1$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Period-t mechanism with $t \neq 1, T$:

Individual i's period-t action space after history $h \in H^t$ such that $g^{-t}(h) \in \mathcal{Y}^{-t}$ is defined by:

$$A_i(h) \equiv \mathcal{D}\left[\Theta | g^{-t}(h), f^{+(t+1)}\right] \times Y^t\left(g^{-t}(h)\right) \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers. Thus, a period-t action of agent i after history $h \in H^t$ consists of an element of the set $Y^t(g^{-t}(h))$, an element of the period-t domain of induced preferences induced by the set Θ at the t-head outcome path $g^{-t}(h)$ and at the socially optimal $t+1$-tail outcome paths $f^{+(t+1)}$, and a nonnegative integer. A typical period-t action played by agent i after history $h \in H^t$ is denoted by $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, (x^t)^i, (z)^i\right)$.

Period-t action space of agents after history $h \in H^t$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-t action profile after history $h \in H^t$.

The period-t outcome function g^t is defined by the following three rules for every $h \in H^t$ such that $g^{-t}(h) \in \mathcal{Y}^{-t}$:

Rule 1: If $a_i(h) \equiv \left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ for every $i \in \mathcal{I}$ and $x^t = \varphi^t\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)$, then $g^t(a(h)) = x^t$.

Rule 2: If $n-1$ agents play $a_j(h) \equiv \left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ with

$$x^t = \varphi^t\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)$$

but agent i plays $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, (x^t)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $x^t R_i\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right](x^t)^i$, then $g^t(a(h)) = (x^t)^i$.
2. If $(x^t)^i P_i\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right] x^t$, then $g^t(a(h)) = x^t$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Period-T mechanism:

Individual i's period-T action space after history $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$ is defined by:

$$A_i(h) \equiv \mathcal{D}\left[\Theta | g^{-T}(h)\right] \times Y^T\left(g^{-T}(h)\right) \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers. Thus, a period-T action of agent i after history $h \in H^T$ consists of an element of the set $Y^T\left(g^{-T}(h)\right)$, an element of the period-T domain of induced preferences induced by the set Θ and the T-head outcome path $g^{-T}(h)$, and a nonnegative integer. A typical period-T action played by agent i after history $h \in H^T$ is denoted by $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right)$.

Period-T action space of agents after history $h \in H^T$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-T action profile after history $h \in H^T$.

The period-T outcome function g^T is defined by the following three rules for every $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$:

Rule 1: If $a_i(h) \equiv \left(R\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ for every $i \in \mathcal{I}$ and $x^T = \varphi^T\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)$, then $g^T(a(h)) = x^T$.

Rule 2: If $n-1$ agents play $a_j(h) \equiv \left(R\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ with $x^T = \varphi^T\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)$ but agent i plays $a_i(h) \equiv \left(\left(R\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $x^T R_i\left[\bar{\theta}|g^{-T}(h)\right]\left(x^T\right)^i$, then $g^T(a(h)) = \left(x^T\right)^i$.
2. If $\left(x^T\right)^i P_i\left[\bar{\theta}|g^{-T}(h)\right] x^T$, then $g^T(a(h)) = x^T$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Let

$$H \equiv \bigcup_{t \in \mathcal{T}} H^t$$

be the set of all possible histories, let $A_i \equiv \bigcup_{h \in H} A_i(h)$ be the set of all actions for agent $i \in \mathcal{I}$, let $A(H)$ be the set of all profiles of actions available to agents, defined by

$$A(H) \equiv \bigcup_{h \in H} A(h),$$

and let $g \equiv \left(g^1, \cdots, g^T\right)$ be the sequence of outcome functions, one for each period $t \in \mathcal{T}$. Note that g satisfies the following properties: a) the outcome function g^1 assigns to period-1 action profile $a(h^1) \in A(h^1)$ a unique outcome in Y^1, and b) for every period $t \neq 1$ and every nontrivial history $h^t \in H^t$, the outcome function g^t assigns to each period-t action profile $a(h^t) \in A(h^t)$ a unique outcome in $Y^t(g^{-t}(h^t))$. Thus, by construction, $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ is a dynamic mechanism.

We now prove that (a) for every $\theta \in \Theta$, there exists a SPE strategy $s^\theta \in S$ of (Γ, θ) such that $g^1(s^\theta(h^1)) = f^1[\theta]$, $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every nontrivial $h^t \in H^t$, and (b) for every $\theta \in \Theta$ and for every $s^\theta \in SPE(\Gamma, \theta)$, $g^1(s^\theta(h^1)) = f^1[\theta]$ and $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every nontrivial $h^t \in H^t$. Thus, fix any state $\theta \in \Theta$.

Let us first prove (a). Since the SCF satisfies the folding condition, we have that $f^1[\theta] = \varphi^1(R[\theta|f^{+2}])$, that $f^t[\theta|g^{-t}(h)] = \varphi^t(R[\theta|g^{-t}(h), f^{+(t+1)}])$ for every nontrivial $h \in H^t$ and every $t \neq 1, T$ and that $f^T[\theta|g^{-T}(h)] = \varphi^T(R[\theta|g^{-T}(h)])$ for every $h \in H^T$.

Let us define agent $i \in \mathcal{I}$'s strategy $s_i^\theta : H \to A_i$ by:

$$s_i^\theta(h^1) = \left(R\left[\theta|f^{+2}\right], \varphi^1\left(R\left[\theta|f^{+2}\right]\right), 0\right),$$

$$s_i^\theta(h) = \left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right], \varphi^t\left(R\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right), 0\right), \quad \text{for every } h \in H^t \text{ with } t \neq 1, T,$$

$$s_i^\theta(h) = \left(R\left[\theta|g^{-T}(h)\right], \varphi^T\left(R\left[\theta|g^{-t}(h)\right]\right), 0\right), \quad \text{for every } h \in H^T.$$

For every period t and history $h^t \in H^t$, to show that $s^\theta|h^t \equiv (s_1^\theta|h^t, \cdots, s_I^\theta|h^t)$ is a SPE of $(\Gamma(h^t), \theta)$ it suffices to show that no agent i can gain by deviating from $s_i^\theta|h^t$ in a single period $\tau \geq t$ and conforming to $s_i^\theta|h^t$ thereafter. To this end, first note that for every history $h \in H$, the strategy profile $s^\theta(h)$ falls into *Rule 1*. Thus, by construction and the fact that the SCF satisfies the folding condition, one can check that $g^1(s^\theta(h^1)) = f^1[\theta]$, $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every $h^t \in H^t$ and every $t \neq 1$.

Fix any period t and any history $h^t \in H^t$. Suppose that agent i deviates from $s_i^\theta|h^\tau$ with $h^\tau \in H|h^t$ by changing only the action $s_i^\theta(h^\tau)$ into $a_i(h^\tau) \in A_i(h^\tau)$. Given that no unilateral deviation from $s^\theta(h^\tau)$ can induce *Rule 3*, the outcome is thus determined by *Rule 2*. But then, under this rule the outcome would only change to be the period-τ outcome announced by this i in her deviation if this outcome is not better than the outcome $g^\tau(s^\theta(h^\tau))$ according to the period-τ induced ordering $R_i[\theta|f^{+2}]$ if $\tau = 1$, to the period-τ induced ordering $R_i[\theta|g^{-\tau}(h^\tau), f^{+(t+1)}]$ if $\tau \neq 1, T$, and to the period-τ induced ordering $R_i[\theta|g^{-\tau}(h^\tau)]$ if $\tau = T$. By noting that $R_i[\theta|f^{+2}]$ is the true period-1 induced ordering of agent i in state θ at the socially optimal 2-tail outcome paths $f^{+2}[\theta|\cdot]$ if $\tau = 1$, that $R_i[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}]$ is the true period-τ induced ordering of agent i in state θ at the head-path $g^{-\tau}(h^\tau)$ and the socially optimal τ-tail outcome paths $f^{+(\tau+1)}[\theta|\cdot]$ if $\tau \neq 1, T$ and that $R_i[\theta|g^{-\tau}(h^\tau)]$ is the true period-τ induced ordering of agent i in state θ at the head-path $g^{-\tau}(h^\tau)$ if $\tau = T$, agent i will not benefit from such a deviation. Since the choice of agent i as well as of the history $h^\tau \in H^\tau|h^t$ are arbitrary, we conclude that the strategy profile $s^\theta|h^t$ is a SPE of $(\Gamma(h^t), \theta)$. Hence, the proposed strategy profile $s^\theta|h$ is a SPE of $(\Gamma(h), \theta)$ for every history $h \in H$, whose outcomes are such that $g^1(s^\theta(h^1)) = f^1[\theta]$, $f^t[\theta|g^{-t}(h^t)] = g^t(s^\theta(h^t))$ for every $h^t \in H^t$ and every $t \neq 1$. This proves our goal (a) stated above. The rest of the proof shows that our goal (b) holds, too.

To see this, assume that the strategy profile s is a SPE of (Γ, θ). Moreover, fix any history $h \in H$. Thus, the strategy profile $s|h$ is a SPE of $(\Gamma(h), \theta)$. Assume, to the contrary, that there is a period $t \in \mathcal{T}$ as well as a history $h^t \in H|h$ such that either $f^t[\theta|g^{-t}(h^t)] \neq g^t(s(h^t))$

if $t \neq 1$ or $f^1[\theta] \neq g^1(s(h^1))$ if $t = 1$. Among all such histories, let $h^\tau \in H|h$ be one of the longest histories. Thus, it must be the case that $f^\tau[\theta|g^{-\tau}(h^\tau)] \neq g^\tau(s(h^\tau))$ and, moreover, that $f^{\hat{\tau}}\left[\theta|g^{-\hat{\tau}}\left(h^{\hat{\tau}}\right)\right] = g^{\hat{\tau}}\left(s\left(h^{\hat{\tau}}\right)\right)$ for every $h^{\hat{\tau}} \in H|(h^\tau, s^\tau(h^\tau))$ if $\tau \neq T$. Note that for the case where $\tau \neq T$ the folding condition mplies that:

$$g^{\hat{\tau}}\left(s\left(h^{\hat{\tau}}\right)\right) = \varphi^{\hat{\tau}}\left(R\left[\theta|g^{-\hat{\tau}}\left(h^{\hat{\tau}}\right), f^{+(\hat{\tau}+1)}\right]\right) \text{ for every } h^{\hat{\tau}} \in H|(h^\tau, s^\tau(h^\tau)) \text{ with } \hat{\tau} \neq T, \text{ and that:}$$

$$g^T\left(s\left(h^T\right)\right) = \varphi^T\left(R\left[\theta|g^{-T}\left(h^T\right)\right]\right) \text{ for every } h^T \in H|(h^\tau, s^\tau(h^\tau)).$$

Also, note that the true profile of period-τ induced orderings at true state θ is:

$$R\left[\theta|f^{+(\tau+1)}\right] \text{ if } \tau = 1,$$

$$R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right] \text{ if } \tau \neq 1, T,$$

$$R\left[\theta|g^{-\tau}(h^\tau)\right] \text{ if } \tau = T.$$

Let us suppose that $\tau \neq 1, T$. Then, the action profile $s(h^\tau)$ is a Nash equilibrium of $\left(\Gamma(h^\tau), R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$.

Suppose that $s(h^\tau)$ falls into *Rule 1* of period-τ mechanism. Thus, $g^\tau(s(h^\tau)) = \varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h), f^{+(\tau+1)}\right]\right)$ for some $\bar{\theta}$, and this outcome is an element of $Y^\tau(g^{-\tau}(h^\tau))$. Since f satisfies the folding condition, an immediate contradiction is obtained if $g^\tau(s(h^\tau)) = \varphi^\tau\left(R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$. Therefore, let us suppose that $g^\tau(s(h^\tau)) \neq \varphi^\tau\left(R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$.

Since f satisfies one-step-ahead Maskin monotonicity and since

$$\varphi^\tau\left(R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right) \neq \varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right),$$

there exists an agent i and a period-τ outcome $y^\tau \in Y^\tau(g^{-\tau}(h^\tau))$ such that

$$\varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right) R_i\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right] y^\tau$$

and

$$y^\tau P_i\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right] \varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right).$$

By changing $s_i(h^\tau)$ into $a_i(h^\tau) = \left(R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right], y^\tau, 1\right)$, agent i can induce *Rule 2* and obtain $g^\tau(a_i(h^\tau), s_{-i}(h^t)) = y^\tau$, thereby contradicting the fact that the action profile $s(h^\tau)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$.

Suppose that $s(h^\tau)$ falls into *Rule 2* of period-τ mechanism. Thus, for every agent $j \neq i$, the period-τ outcome determined by this rule is maximal for this j in $Y^\tau(g^{-\tau}(h^\tau))$ according to her period-τ induced ordering $R_j\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]$. Moreover, given that the action profile $s(h^\tau)$ is a Nash equilibrium of $\left(\Gamma, R\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$, for agent i it holds that the outcome $g^\tau(s(h^t))$ is such that $g^\tau(s(h^t))$ is an element of the weak lower contour set of $R_i\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]$ at $\varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$ and that

$$g^\tau\left(s\left(h^t\right)\right) R_i\left[\theta|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right] x^\tau$$

for every x^τ in the weak lower contour set of $R_i\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]$ at $\varphi^\tau\left(R\left[\bar{\theta}|g^{-\tau}(h^\tau), f^{+(\tau+1)}\right]\right)$.

Since the SCF f satisfies the one-step-ahead weak no veto-power, this implies that

$$g^{\tau}\left(s\left(h^{\tau}\right)\right)=\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right),f^{+(\tau+1)}\right]\right).$$

The folding condition implies that $\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right),f^{+(\tau+1)}\right]\right)=f^{\tau}\left[\theta|g^{-\tau}\left(h^{\tau}\right)\right]$, which is a contradiction.

Suppose that $s\left(h^{\tau}\right)$ falls into *Rule 3* of period-τ mechanism. Thus, for every agent j, the period-τ outcome determined by this rule is maximal for this j in $Y^{\tau}\left(g^{-\tau}\left(h^{\tau}\right)\right)$ according to her period-τ induced ordering $R_j\left[\theta|g^{-\tau}\left(h^{\tau}\right),f^{+(\tau+1)}\right]$. Since the SCF f satisfies the one-step-ahead unanimity, we have that $g^{\tau}\left(s\left(h^{\tau}\right)\right)=\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right),f^{+(\tau+1)}\right]\right)$. The folding condition implies that $\varphi^{\tau}\left(R\left[\theta|g^{-\tau}\left(h^{\tau}\right),f^{+(\tau+1)}\right]\right)=f^{\tau}\left[\theta|g^{-\tau}\left(h^{\tau}\right)\right]$, which is a contradiction.

We conclude the proof by mentioning that, suitably modified, the above proof provided for the case where $\tau\neq 1,T$ applies to the case where $\tau=1$ as well as to the case where $\tau=T$. ■

Proof of Lemma 1

Proof of Lemma 1. Let the premises hold. Fix any $x^1\in X^1$ and any $\theta\in\Theta$. Let $x^2\left[\eta|x^1\right]$ be the solution to:

$$\frac{\partial U(\eta,x^1,x^2)}{\partial x^2}=0.$$

By the implicit function theorem, we have that:

$$\frac{\partial x^2\left[\eta|x^1\right]}{\partial\eta}=-\frac{\frac{\partial^2 U\left(\eta,x^1,x^2\left[\eta|x^1\right]\right)}{\partial^2 x^2}}{\frac{\partial^2 U\left(\eta,x^1,x^2\left[\eta|x^1\right]\right)}{\partial\eta\partial x^2}}>0.$$

Therefore, the peak for the median type $\eta=\theta_{med}$ is always the peak in the second voting stage for each $x^1\in X^1$. Write $x^2\left[\theta_{med}|x^1\right]$ for the peak of the median type in the second voting stage conditional on x^1.

Since it holds that:

$$\frac{\partial U\left(\theta_{med},x^1,x^2\left[\theta_{med}|x^1\right]\right)}{\partial x^2}=0,$$

from the implicit function theorem we obtain that:

$$\frac{\partial x^2\left[\theta_{med}|x^1\right]}{\partial x^1}=-\frac{\frac{\partial^2 U\left(\theta_{med},x^1,x^2\left[\theta_{med}|x^1\right]\right)}{\partial x^1\partial x^2}}{\frac{\partial^2 U\left(\theta_{med},x^1,x^2\left[\theta_{med}|x^1\right]\right)}{\partial^2 x^2}}\geq 0.$$

Let us show that $x^2\left[\theta_{med}|x^1\right]$ is the Condorcet winner under $R\left[\theta|x^1\right]$ for every $x^1\in X^1$. For every allowable type $\eta\in\left(\underline{\eta},\bar{\eta}\right)$ and policy (x^1,x^2), let:

$$\Phi(\eta,x^1,x^2)=U\left(\eta,x^1,x^2\left[\theta_{med}|x^1\right]\right)-U(\eta,x^1,x^2).$$

Then, for every $x^2 < x^2\left[\theta_{med}, x^1\right]$, we have that:

$$\Phi(\theta_{med}, x^1, x^2) = \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial U(\theta_{med}, x^1, z^2)}{\partial z^2} dz^2.$$

Furthermore, for every $\eta > \theta_{med}$, it holds that:

$$\Phi(\eta, x^1, x^2) - \Phi(\theta_{med}, x^1, x^2) = \int_{\theta_k}^{\eta} \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial^2 U(\alpha, x^1, z^2)}{\partial \alpha \partial z^2} dz^2 d\alpha > 0.$$

Since

$$\Phi(\theta_{med}, x^1, x^2) = U(\theta_{med}, x^1, x^2\left[\theta_{med}|x^1\right]) - U(\theta_{med}, x^1, x^2) \geq 0,$$

it follows that:

$$\Phi(\eta, x^1, x^2) > 0,$$

which, in turn, guarantees that:

$$U(\eta, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\eta, x^1, x^2).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, it holds that:

$$U(\theta_j, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\theta_j, x^1, x^2).$$

Likewise, for every $x^2 > x^2\left[\theta_{med}|x^1\right]$, one can show that for every voter $j = 1, \cdots, k-1$ it holds that:

$$U(\theta_j, x^1, x^2\left[\theta_{med}|x^1\right]) > U(\theta_j, x^1, x^2).$$

Therefore, $x^2\left[\theta_{med}|x^1\right]$ is a Condorcet winner under $R\left[\theta|x^1\right]$, that is, $CW\left(R\left[\theta|x^1\right]\right) = x^2\left[\theta_{med}|x^1\right]$, and so the majority voting function $f^2\left[\cdot|\cdot\right]$ is a single-valued function for every $\theta \in \Theta$ and every $x^1 \in X^1$.

Let $x\left[\theta_{med}\right] = \left(x^1\left[\theta_{med}\right], x^2\left[\theta_{med}\right]\right)$ be the global peak for the median type θ_{med}. Next, we show that $x^1\left[\theta_{med}\right]$ is the Condorcet winner under $R\left[\theta|f^2\right]$.

Solving backward, given that the majority voting function $f^2\left[\theta|x^1\right] = x^2\left[\theta_{med}|x^1\right]$ for every $x^1 \in X^1$, we have that the reduced utility of type η is:

$$V(\eta, x^1) = U(\eta, x^1, x^2\left[\theta_{med}|x^1\right]).$$

Then, we have that:

$$\frac{\partial V(\eta, x^1)}{\partial x^1} = \frac{\partial U(\eta, x^1, x^2\left[\theta_{med}|x^1\right])}{\partial x^1} + \frac{\partial U(\eta, x^1, x^2\left[\theta_{med}|x^1\right])}{\partial x^2} \frac{\partial x^2\left[\theta_{med}|x^1\right]}{\partial x^1},$$

and so, by Definition 12, it follows that:

$$\frac{\partial^2 V(\eta, x^1)}{\partial \eta \partial x^1} = \frac{\partial^2 U(\eta, x^1, x^2\left[\theta_{med}|x^1\right])}{\partial \eta \partial x^1} + \frac{\partial^2 U(\eta, x^1, x^2\left[\theta_{med}|x^1\right])}{\partial \eta \partial x^2} \frac{\partial x^2\left[\theta_{med}|x^1\right]}{\partial x^1} > 0.$$

Then, for every $x^1 \in X^1$, let:

$$\Pi(\eta, x^1) = V(\eta, x^1[\theta_{med}]) - V(\eta, x^1)$$

Next, take any $x^1 < x^1[\theta_{med}]$. Then, it holds that:

$$\Pi(\theta_{med}, x^1) = \int_{x^1}^{x^1[\theta_{med}]} \frac{\partial V(\theta_{med}, z^1)}{\partial z^1} dz^1.$$

Moreover, for every $\eta > \theta_{med}$, it also holds that:

$$\Pi(\eta, x^1) - \Pi(\theta_{med}, x^1) = \int_{\theta_{med}}^{\eta} \int_{x^1}^{x^1[\theta_{med}]} \frac{\partial^2 V(\alpha, z^1)}{\partial \alpha \partial z^1} dz^1 d\alpha > 0.$$

Since

$$\Pi(\theta_{med}, x^1) = V(\theta_{med}, x^1[\theta_{med}]) - V(\theta_{med}, x^1) \geq 0,$$

we have that:

$$\Pi(\eta, x^1) > 0,$$

which, in turn, guarantees that:

$$V(\eta, x^1[\theta_{med}]) > V(\eta, x^1).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, we have that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1).$$

Likewise, for every $x^1 > x^1[\theta_{med}]$ one can also show that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1), \quad \text{for every voter } j = 1, \cdots, k-1.$$

We conclude that $x^1[\theta_{med}]$ is a Condorcet winner under $R[\theta|f^2]$, that is, $CW(R[\theta|f^2]) = x^1[\theta_{med}]$, and so the majority voting function $f^1[\cdot]$ is a single-valued function for every $\theta \in \Theta$.
■