Stable jurisdiction partitions under monotonically decreasing population density*

Working paper

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Abstract

Starting from Alesina-Spolaore breakthrough publication, many papers study the problem of jurisdiction formation and spatial allocation of public goods. Bogomolnaia, Le Breton, Savvateev and Weber (2007) demonstrated that there may be no coalitionary stable partitions in one-dimensional world under a specifically chosen population density. This work constructs two new counterexamples: for a single-peaked density function and for a monotonically decreasing density function. On the other hand it is shown that adding certain "smoothness" condition yields existence theorem for monotonically decreasing density. A stable partition is given by a simple and natural construction. The sizes and populations of jurisdictions in this partition are analyzed. It is found that they are changing monotonically and under certain conditions satisfy Zipf's "rank–population" law.

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1 Introduction

In various frameworks of coalition formation there is a trade-off between cost efficiency and homogeneity of a group. Suppose that a group provides a club good. On the one hand members of a small group bear a higher fraction of the fixed cost. On the other hand a small and homogeneous group may choose characteristics of the good to better satisfy their needs and tastes. We provide three specific examples of such "horizontal differentiation".

The first story, geographical. Suppose that a group of agents are distributed at some space and want to get a public service at some facility (school, clinic, gym, cinema etc.) The cost of building and maintaining a facility has a fixed component divided among the users. The more agents use the same facility the less share accrues to each user. At the same time the location of the facility is fixed and the more agents use it the further some of them should commute to reach the facility. Thus there is a trade-off between economy of scale and transportation efficiency. If there are too many facilities then some of them could be abandoned with users transferring to other facilities. If there are too few facilities then new facilities could be formed. An issue of stability arises: does a stable configuration always exist and in what particular sense?

One common interpretation of this story is political: a public service is interpreted as a government or a parliament, and a group as a nation, region, municipality etc. The government is located at the capital and the citizens need to commute there to get some public services. Possible instabilities include unification, dissolution, border changes, secession or moving of capital. Many examples of such processes took place in recent years around the world and much more proposals are being discussed. Is it possible to draw stable boundaries?

The second story, political. Now we consider a virtual space of political convictions. Suppose that a number of politicians are forming several parties. On the one hand the bigger the party the better for its members. Firstly, a political party may have some fixed costs divided among its members. Secondly, a larger party has better electoral perspective and is more influential. Sometimes there is a threshold on headcount to obtain an official license. On the other hand a large party is subject to internal tensions. The party program is an outcome of many compromises and a particular member may be not very happy with the result. Thus we have a similar trade-off between size and homogeneity. Some political systems are stable and has a low number of parties (For instance, two major and several minor parties, like in the

U.S.). In the others there is a plenty of parties forming tactical unions with new projects being created before every election. A good example is modern Ukraine with 47 parties represented in local councils. Does the process always converge to a steady state?

The third story, customer. Here we also analyze a virtual space of consumer characteristics. Some goods are better to be consumed in groups, probably the most convenient example is a sightseeing or hiking tour. But members of a group may have different attitudes towards the tour duration, intensity and other features. If a group is small and homogeneous then it can choose the most appropriate tour characteristics but each member bear a larger share of fixed cost (bus rent, carrying camp equipment etc.). If a group is large then the fixed cost shares are small but many members are not satisfied with the chosen tour specification. Thus we face the same trade-off between size and homogeneity. Is it always possible to divide a society of tourists into stable groups?

We proceed in the tideway of traditional model presented in papers [Alesina and Spolaore, 1997] and [Bogomolnaia et al., 2007]. Namely, the space is one-dimensional and bounded, agents live across the segment continuously according to some density. The capital of a jurisdiction is located at its median. Public good procurement costs are divided equally whereas transportation costs are born individually. Utility is not transferable. The notion of stability is a coalitional one [Aumann and Dréze, 1974]: a jurisdiction partition is stable if no group has incentives to secede and form a new jurisdiction, thus decreasing the costs of each member.

It is shown in [Alesina and Spolaore, 1997] that a stable partition always exists under the uniform density. We study the case of a non-uniform density. It is shown in [Bogomolnaia et al., 2007] that no stable partition may exist in the discrete model. We restate this result in the continuous model and find two more specific counterexamples and sufficient conditions for existence of a stable partition. More specifically, we show that a counterexample may be designed for a single-peaked and even monotonic population density. But adding a mild "smoothness" condition guarantees that a stable partition exists in the monotonic case. One stable partition is constructed in an explicit way and we show that it demonstrates the following behavior:

- The population of jurisdictions monotonically decrease;
- If the initial distribution satisfies the power law then the jurisdiction populations also satisfy power law.

The last observation gives a possible way to establish the well-known Zipf's law. (See [Zipf, 1949, Gabaix, 1999, Nitsch, 2005] among others).

The rest of the text is organized in the following way. Section 2 presents the formal model. In section 3 we formulate general results and give the proof ideas. Section 4 concludes.

2 The model

We describe a society split into jurisdictions. Each jurisdiction locates a public facility and finance it. Members of the jurisdictions face two types of cost. First, they share financing of the facility with other members of their jurisdictions. Second, they bear the transportation cost. We assume that the society consists of individuals distributed along a segment $\Omega = [0, M]$. The housing density function $f(x)$ determines the mass $\mu(S) = \int_S f(x)\,dx$ of the population located in a Lebesgue measurable subset $S \subset \Omega$.

We also assume that all members of the jurisdiction finance the facility equally, and the total cost of the facility is independent of its location and number of users. Without loss of generality, 1 is assigned to this total cost. Then each member of S pays $1/\mu(S)$ in order to finance the facility. The notion of transportation cost involves individuals' location. For the sake of simplicity we identify each individual with her location. Let m be the location of the facility chosen by a jurisdiction S. Then the transportation cost faced by a member $x \in S$ of the jurisdiction S is defined as the distance $|x - m|$ between this individual and the facility financed by the jurisdiction S. Finally, the total cost that individual $x \in S$ incurs is equal to

$$C_x(S, m) = \frac{1}{\mu(S)} + |x - m|. \tag{1}$$

In the case of $\mu(S) = 0$ the cost is infinite. It means that leaving an arbitrary atomic jurisdiction, an individual does not increase her cost.

In order to minimize the aggregate transportation cost of its members a non-empty jurisdiction S locates the facility into a median $m(S)$ of S [Haimanko et al., 2004]. The set $\mathrm{med}(S)$ of the medians can consist of more than a single point if S is disconnected. If the median $m(S)$ is unique we drop the dependence of the total cost on m and write $C_x(S)$ because the median m is completely defined by S.

We define a *jurisdiction partition* Σ as a system of sets $\{S_i, i \in I\}$ and medians $m_i \in \mathrm{med}(S_i)$, such that I is a set of indices, $\bigcup_{i \in I} S_i = \Omega$, and $\mu(S_i \cap S_j) = 0$ if $i \neq j$. In other words, the society Ω is split into mutually disjoint jurisdictions $\{S_i\}$, and the median m_i of each jurisdiction is chosen. An individual x assigned to a jurisdiction S_i incurs the cost

$$C_x(\Sigma) = C_x(S_i, m_i) = \frac{1}{\mu(S_i)} + |x - m_i|. \tag{2}$$

Given a jurisdiction partition, some individuals could be prone to exit the jurisdictions, which they are assigned to, and form a new jurisdiction. We address the question of the jurisdiction partition stability under such group deviations according to [Aumann and Dréze, 1974]. More precisely, a measurable group $T \subset \Omega$ *blocks* partition Σ, if for some median $m^* \in \mathrm{med}(T)$ and every agent $x \in T$ it holds that $C_x(T, m^*) \leq C_x(\Sigma)$ and for some agent the inequality is strict. We call a partition Σ *stable*, if no group blocks it.

In the following analysis we will consider only partitions into *connected*, or *stratified* jurisdictions. It means that there is a sequence of border points $y_0 = 0, y_1, y_2, \ldots, y_n = M$ and a jurisdiction S_i is defined as $[y_{i-1}, y_i]$.

3 General results

In this section we formulate and prove main results regarding the existence of a stable jurisdiction partition.

3.1 Formulations of theorems

We introduce two assumptions regarding the population density. They exhibit behavior of the population density on the segment $\Omega = [0, M]$.

Assumption 1. *The population probability density $f(\cdot)$ is continuously differentiable and single-peaked on the segment Ω.*

Assumption 2. *The population probability density $f(\cdot)$ is continuously differentiable and strictly decreasing on the segment Ω.*

Clearly, the second Assumption is a particular case of the first one. First we claim that Assumption 1 is too weak to provide the stability:

Theorem 1. *There exists a probability density function satisfying Assumption 1 such that an arbitrary partition of Ω into connected jurisdictions is unstable.*

Theorem 1 continues a list of negative results established by [Bogomolnaia et al., 2007, Savvateev, 2012].

Even Assumption 2 still is not enough.

Theorem 2. *There exists a probability density function satisfying Assumption 2 such that an arbitrary partition of Ω into connected jurisdictions is unstable.*

Finally, we introduce another rather technical assumption.

Assumption 3. *The population probability density $f(\cdot)$ is continuously differentiable and satisfies condition*

$$|f'(x)| \leq 2\sqrt{2}f(x)^{3/2}.$$

This condition means that the density function changes "smoothly", without sharp variations. It can be also shown that "smoothness" and monotonicity imply certain single-crossing condition.

Theorem 3. *Suppose that Assumptions 2 and 3 holds. Then a monotonically increasing sequence $y_0 = 0$, y_1, y_2, $\ldots$, $y_n = M$, exists such that the partition $\bigcup_i S_i$, $S_i = [y_{i-1}, y_i)$, of Ω is stable.*

The sketches of the proofs are presented in the following subsections.

3.2 Non-existence result for the single-peaked case: proof idea

We construct a single-peaked population density function such that any partition into connected jurisdictions is unstable. The density function is symmetric and consists of constant and inverse quadratic segments. It means that $f(x)$ is equal either to some constant c or to $\frac{k}{x^2}$ for some constant k. The latter function has a nice property: if k is small enough then the leftmost member of a jurisdiction bears total cost at least βx for some $\beta > 1$. For clarity, we construct a discontinuous function, but it can be approximated by a continuous one without affecting the proof. Also we provide a construction for an infinite world $\Omega = (-\infty, \infty)$ but it also works for all sufficiently large finite symmetric worlds $[-M, M]$. Transforming to $[0, M]$ is straightforward.

The density function is constructed in the following manner. Firstly, there is a small but very densely populated "core" $[-\epsilon, \epsilon]$ for some small ϵ. Let the population density there be $\frac{100}{\epsilon^2}$. For this density there must be several jurisdictions lying entirely in the core and bearing very little monetary cost. If an agent located at $x > \epsilon$ joins such coalition then she would bear cost at most $x + \epsilon$. Secondly, there are moderately populated "urban territories" $[-z, -\epsilon] \cup [\epsilon, z]$ on both sides of the core. The density there decreases quadratically: $\frac{3}{x^2}$. Finally, there are "rural areas" $(-\infty, z] \cup [z, \infty)$ behind the edges of urban territories with density $\frac{2.7}{x^2}$. The parameters are chosen such that the following conditions hold:

(i) If $|x| > \epsilon$ then an agent at point x bears cost greater than $1.2x$ being attached to an infinite jurisdiction $[x, +\infty)$;

(ii) For some $x_0 > 1000\epsilon$ and $y_0 > x_0$ an agent at point x_0 bears cost less than $0.99x_0$ in jurisdiction $[x_0, y_0]$;

(iii) For all sufficiently large x_1 and all $y_1 > x_1$ an agent at point x_1 bears cost greater than $1.01x$ in jurisdiction $[x_1, y_1]$.

The proof proceeds in the following steps: firstly, in a stable partition there should be several jurisdictions inside the core. Secondly, an infinite jurisdiction $[\delta, +\infty)$ is unstable for $\delta < 5\epsilon$. Indeed, it can be shown that its median is less than 10ϵ. It means that an agent located at $x_0 > 10\epsilon$ pays at least $x_0 - 10\epsilon$. If $x_0 > 1000\epsilon$ then $x_0 - 10\epsilon > 0.99x_0$. It means that if x_0 is in accord with condition (ii), then the agent located at x_0 is better off in the coalition $[x_0, y_0]$. All other members of this coalition are better off all the more, so the coalition is blocking. Thirdly, if there is an infinite jurisdiction $[x, +\infty)$ for $x > 5\epsilon$, then the agent located at x bears cost at least $1.2x$ according to condition (i). This is greater than $x + \epsilon$, so this agent has an incentive to join the central coalition. Fourthly, if there are no infinite jurisdictions then all jurisdictions have the form $[x_1, y_1]$, where x_1 can be arbitrarily large. In particular, it may satisfy condition (iii). In this case an agent at x_1 bears cost at least $1.01x_1$ that is greater than $x_1 + \epsilon$. Thus, in any case some distant agents wish to join the central coalition. Finally, we note that the same holds for the negative half of the line. If two equal segments join the same central coalition then its median remains unchanged and the old members also win from expanding. Thus the central jurisdiction plus two distant segments form a blocking (disconnected) coalition.

3.3 Non-existence result for the decreasing case: proof idea

The previous argument relies crucially upon the symmetry of housing density. This guarantees that agents wishing to join a central coalition will be admitted. In the decreasing density framework we need another effect. We start by considering an illustrative example.

Suppose that $\Omega = [0, 4.6]$ and the density is the following:

$$f(x) = \begin{cases} 101, & x \in [0, 0.05]; \\ 1, & x \in [0.06, 4.6]; \\ 1 + 10000(0.06 - x), & x \in [0.05, 0.06]. \end{cases}$$

This density is characterized by a sharp decline at 0.05. The parameters are chosen such that the median of Ω also lies at 0.05. It can be shown that $C_0([0, x])$ is monotonically decreasing in x. That is, the agent 0 is the best off at the grand coalition $[0, 4.6]$. But this coalition is unstable: for instance, the coalition $[2.6, 4.6]$ is blocking. Indeed, their costs are at least 2.5 in the grand coalition and at most 1.5 after seceding.

Thus, in any stable partition the leftmost coalition wishes to expand towards the right edge. This effect is used instead of symmetric expansion in a construction similar to previous one. We proceed by describing a full counterexample that yields theorem 2. As before, we give a proof for an infinite world, but it is also valid for sufficiently large finite world. Specifically, it is convenient to consider $\Omega = [0.9\epsilon, +\infty)$. The population density is the following:

$$f(x) = \begin{cases} \frac{100}{\epsilon^2}, & x \in [0.9\epsilon, \epsilon]; \\ \frac{3}{x^2}, & x \in [\epsilon, z]; \\ \frac{2.7}{x^2}, & x \in [z, +\infty). \end{cases}$$

Thus the world is again divided into the core, "urban" and "rural" territories but now the core is too small to host an optimal jurisdiction. Moreover, the optimal coalition for the border agent at 0.9ϵ is the grand coalition. But conditions (i)–(iii) from the previous section still hold. Condition (ii) guarantees that the grand coalition is unstable. Repeating the line of reasoning from subsection 3.2 we deduce that in any stable partition some agents would like to join the leftmost coalition. But since its population is less than optimal they would agree to admit the newcomers. Thus, there always exists a blocking coalition and no stable partition could be found.

3.4 Existence result: the construction and its validity

In this subsection we construct a stable jurisdiction partition for theorem 3 and argue for its validity.

A stable construction is designed in a very natural way. Consider the leftmost agent $y_0 = 0$ and find a jurisdiction $[y_0, y_1]$ that is the best for her. Then consider the new leftmost agent y_1 and find a jurisdiction $[y_1, y_2]$ optimal to her. This process is repeated until every agent is attached to some distribution.

In general the described partition may not be stable as we have shown in subsection 3.3. But Assumption 3 guarantees stability. The proof idea is the following: the leftmost agent of a potentially blocking coalition would not agree to secede. Indeed, define $C_y(\Sigma)$ as the total cost in the designed partition and $C^{\min}(y) = \min_z C_y([y, z])$ as the minimal possible cost of agent y in a coalition she is the leftmost member of. It is sufficient to show that

$$C^{\min}(y) > C_y(\Sigma) \tag{3}$$

for all y except border ones. This inequality is clear if y lies to the left of the median. Indeed, $C_y(\Sigma)$ decreases: the monetary cost is the same and the transportation cost is decreasing. On the other hand, $C^{\min}(y)$ increases: for any fixed population the size of the jurisdiction and the distance to the median increase while moving rightwards. Thus the same holds for the optimal population. Since $C^{\min}(y) = C_y(\Sigma)$ for the left border we obtain the necessary inequality.

Inequality (3) is also clear if y is close to the right edge of the jurisdiction. Indeed, $C_{y_i}([y_{i-1}, y_i])$ is less than $C_{y_i}([y_i, y_{i+1}])$. Again this is true for any fixed population and thus must be true for the optimal ones. Thus the problem may arise only when y lies in the middle part of the right half. But Assumption 3 guarantees that it does not arise. The intuition behind is that it "takes a long distance" for $C_y(\Sigma)$ to "catch" $C^{\min}(y)$ after the advantage received in the left half. This distance is on hand only if the density in the left half differs much from the density in the right half. It may happen only if the density changes sharply at some point and it is prohibited by Assumption 3. Some special treatment is needed on the right edge where the optimal jurisdiction is defined by a corner solution instead of inferior one.

The argument is illustrated on figure 1.

3.5 Zipf's law

We have already shown that in our stable partition the population of the jurisdiction decreases from left to right. Thus the ordinal number of a jurisdiction when counting from left to right and its rank by population coincide. The well-known Zipf's law establishes a dependency between

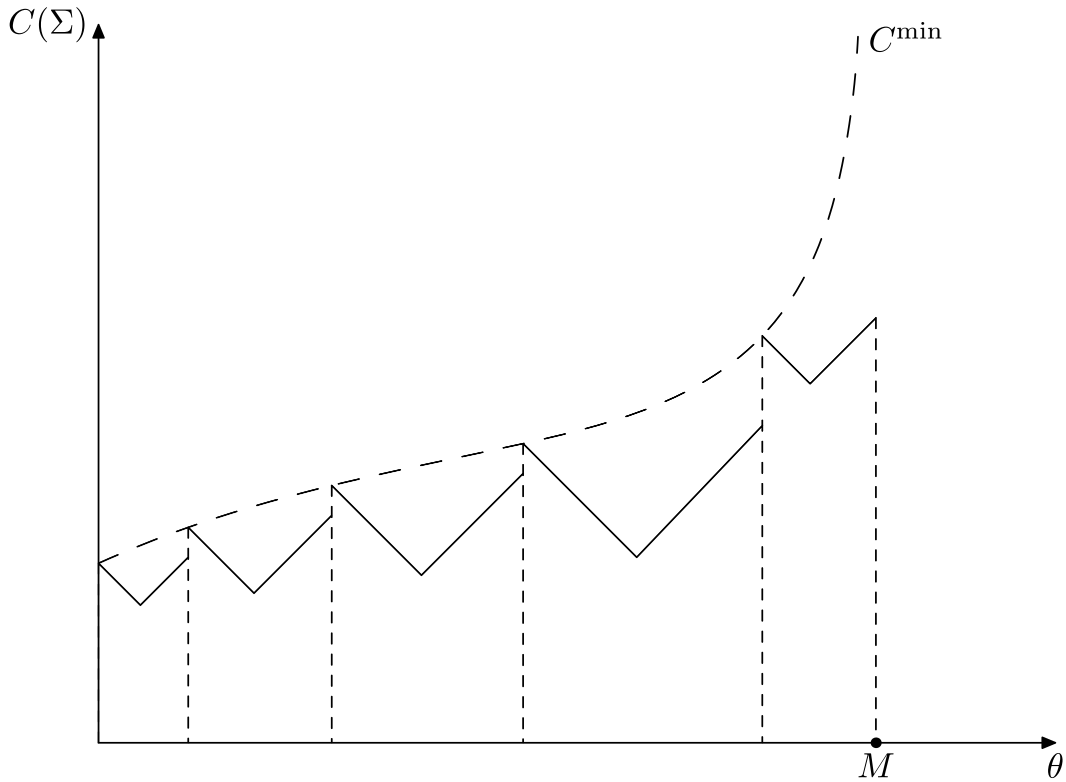

Figure 1: The constructed partition and costs.

population and rank: $P_n \sim n^{-\beta}$. Our calculations show that this is indeed the case in our model if the initial population density does also satisfy the power law. Specifically we prove the following theorem:

Theorem 4. *If population density $f(x)$ is proportional to $x^{-\gamma}$ and a fairly large number of jurisdictions is formed according to our construction then for sufficiently large n it holds that*

$$P_n \approx \frac{\Lambda_\gamma}{n^\beta}, \quad \beta = \frac{2-\gamma}{\gamma}, \tag{4}$$

where Λ_γ is a constant that depends only on the parameters of the distribution.

4 Conclusion

The problem of jurisdiction formation is still of current interest. We proved that under monotonically decreasing and smoothly changing population density a stable jurisdiction partition always exists. Our counterexamples are very specific, with carefully tuned parameters. Apparently, a stable partition may exist for much broader class of densities, that might approximate the actual housing density. If it is so, real jurisdiction formation may converge to some stable partition as usually occurs within a country. Unfortunately, separatist movements often relegate economic concerns to the background, and polarized human convictions come to the front. In this case there may be no stable configuration and jurisdiction formation should be constrained.

From the theoretical point of view there are two interesting questions for future research. Firstly, suppose that a coalition blocks a partition only if all its members substantially decrease their costs. Does there always exist such an approximate equilibrium and for which values of parameters? Secondly, what is the computational complexity of determining whether an equilibrium exists and of finding it? Is this problem complete for any complexity class?

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