Endogenous Formation of Multiple Social Groups

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Abstract

In this paper, we propose a model of group formation to analyze the individual decision to join or leave multiple social groups. The stability and the efficiency of bipartite graphs between agents and social groups they participate in are characterize through not only a central agent (the grand star) but also a set of semi-central agents (mini stars). In equilibrium, a connected graph which contain a grand star or mini stars, under several conditions, could become an efficient graph.

Keywords: social group, bipartite graph formation, efficiency, stability.

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1 Background

In the literature on graph theory, applications of bipartite graphs to social sciences are wide and varied, generally separated into two main types. The first type includes social bipartite graphs between two independent types of entities such as buyers and sellers, or online users and objects, e.g. music, movies, bookmarks (see, for example, Zhang, Zhang, and Liu, 2013; Lambiotte and Ausloos, 2005; Dahui, Li, and Zengru, 2006; Kranton and Minehart, 2001; Wang and Watts, 2006; Corominas-Bosch, 2004, 1999). The second type covers social bipartite graphs in which agents belong to particular groups, for example, directors and the board of directors, firms and the markets they operate in, agents and public good groups (Battiston and Catanzaro, 2004; Mizruchi, 1996; Vallée and Massol, 2013; Koskinen and Edling, 2012; Page Jr. and Wooders, 2010; Richefort, 2018). In this second type of social bipartite graphs, agents belong to one set of entities, and social groups are to another one.

The concept of social bipartite graphs have been discussed in two streams of the literature. Firstly, bipartite graphs have been considered in the area of game theory, such as analyses of the bilateral bargaining process between traders or the effect of exogenous graph structures on cooperation and economic outcomes in equilibrium (see, for example, Kranton and Minehart, 2001; Wang and Watts, 2006; Corominas-Bosch, 2004, 1999). Secondly, researchers have focused on analyzing structural properties of bipartite graphs. In real world, there are several common properties of bipartite graphs such as Small World effect, highly cluster, assortative, and dominated by a giant component (see, for example, Battiston and Catanzaro, 2004; Davis et al., 2003; Newman and Park, 2003; Newman et al., 2001). However, degree distribution of sets of nodes are case specific. Comparing the bipartite cooperation in the ecology and in the garment industry, Saavedra et al. (2008) showed that both networks have similar structure patterns, and produce exponential degree distributions for two set of nodes. Studying the relationship between music listeners and music groups, Lambiotte and Ausloos (2005) showed that, while the degree distribution of listeners follows power-law, the shape of music groups is determined by an exponential degree distribution. By contrast, Zhang et al. (2013) stated that the degree distribution of users in online bipartite graphs actually follows the Mandelbrot's law and the degree distribution of objects such as music, bookmarks, or movies are power-law. Considering bipartite producer-consumer networks, Dahui et al. (2006) concluded that the producers' degree distribution depends on the value of a uniform initial attractiveness.

Despite of a large number of studies on bipartite graphs, there is a lack of analysis on the bipartite graph formation mechanism, particularly between agents and social groups that they may participate in. Still, researchers have applied game theory, including both cooperative and non-cooperative games, for studying group formation and competition among groups (see, for example, Arnold and Wooders, 2005; Hart and Kurz, 1983; Bloch, 2005).[1] But, to our knowledge, only Page Jr. and Wooders (2010) modelized group formation using bipartite directed graph structures in a non-cooperative game to discuss existence of a Nash equilibrium in the bipartite graph.

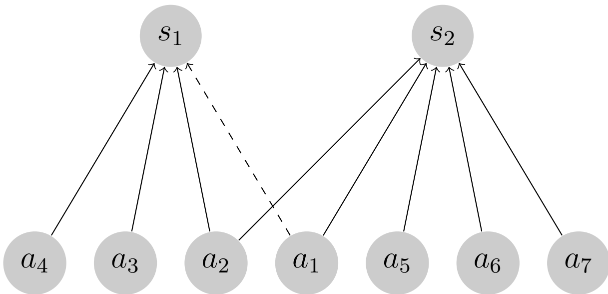

Figure 1: An example of a bipartite graph structure of two social groups

In this paper, we propose a model of group formation to analyze the individual decision to join or leave multiple social groups. Nowadays, social groups play an important role as being a member of a social group provides the chance to access useful resources which are only shared among its members. From an individual point of view, both direct and indirect benefits can be obtained from participation in a social group. For instance, considering a bipartite graph structure of two social groups as in Figure 1, if an agent a_1 is a member of a social group s_1, she obtains a direct benefit from her membership. However, if a_1 is not a member of s_1, she may still obtain a benefit from s_1 provided that she knows a member of s_1. More precisely, a_1 may obtain an indirect benefit from s_1 if she participates in another group s_2 where one of the members, say agent a_2, is also a member of s_1. Hence, a_1 and a_2 know each other through s_2. In other words, s_1 and s_2 are connected thanks to agent a_2, which allow the members of s_2 (including agent a_1) to receive an indirect benefit from s_1 and vice versa. As a consequence, the decision of a_1 to participate in s_1 or not should depend on the value of both direct and indirect benefits she could get from s_1.

So, the purpose of this paper is to characterize efficient and stable membership structures in the presence of many social groups. To this end, we

[1]See Bloch (2005) for a review of group and network formation models to Industrial Organization, mainly focusing on the formation of conclusive agreements, cost-reduced alliances, and trade networks.

use the definition of strong efficiency, also used in (Jackson and Wolinsky, 1996), which assumes that a bipartite graph is efficient when it maximizes social welfare. Regarding the notion of stability, we assume that agents can unilaterally form or sever links. Consequently, a bipartite graph between agents and social groups is stable if no members want to leave out groups they participate in, and no outsiders want to join any group. These two conditions are identical to the internal condition and the external condition, respectively, developed in open membership games (Selten, 1973; d'Aspermont et al., 1983).

We derive our results using two new concepts: a grand star and a mini star. A grand star is an agent who simultaneously participates in all social groups, and thus, connects all agents together in society. A mini star is an agent who simultaneously participates in several but not all social groups, and thus, connects some communities of agents together.[2] We obtain conditions under which a connected bipartite graph which contains a grand star or a set of mini stars is efficient. Likewise, stable bipartite graphs are characterized through not only one central agent (the grand star) but also a set of semi-central agents (mini stars). Finally, we show that there exist bipartite graph structures which are simultaneously stable and efficient, whatever the number of agents and the number of social groups.

The remainder of the paper is organized as follows. Section 2 set up our model of social group formation. Sections 3 analyzes the efficiency and stability of bipartite graphs in the two social groups formation model. Section 4 analyzes the case of three social groups as a representative example of the generalized social groups formation model which discussed in the last section.

2 A model of social groups formation

There are m social groups, $s_1, \ldots, s_m$, and n agents, $a_1, \ldots, a_n$. Each agent a_i has to decide which social groups to join and belong to. The membership formation process produces a graph structure, formally represented by a bipartite graph where social groups are listed on one side and agents are listed on the other side.

More precisely, the result of the membership formation process is formalized as a triplet $g = (A, S, L)$, where $A = \{a_1, ..., a_n\}$ and $S = \{s_1, ..., s_m\}$ are two disjoint sets of nodes formed by agents and social groups, and L is

[2]Note that a star in our analysis is different from the star in the study of Jackson and Wolinsky (1996). While a grand star or mini stars are agents who connect social groups together, the star in Jackson and Wolinsky (1996) is a graph in which all agents are linked to one central agent and there are no other links.

a set of links, each link joining an agent with a social group. A link between agent a_i and social group s_j means that a_i is a member of s_j, and is denoted as ij. Let $g+ij$ denote the bipartite graph obtained by adding link ij to the set of links already in g, and let $g-ij$ denote the bipartite graph obtained by deleting ij from the set of links already in g.

A bipartite graph is connected if there exists a path linking any two nodes of the two disjoint sets A and S. A path in g between two nodes, say a_i and s_j, is a set of distinct nodes $\{a_{i_1}, s_{j_1}, a_{i_2}, s_{j_2}, \ldots, s_{j_{k-1}}, a_{i_k}, s_{j_k}\}$ among $A \cup S$ such that:

$$\{i_1 j_1, i_2 j_1, i_2 j_2, \ldots, i_k j_{k-1}, i_k j_k\} \subseteq L,$$

with $i_1 = i$ and $j_k = j$. A cycle in g is a path that starts and ends at the same node. A bipartite graph is acyclic if it contains no cycle.

Let b_j denote the intrinsic value of being a member of social group s_j and c_{ij} denote a_i's cost of membership in social group s_j. The utility function $u_i : g \to \mathbb{R}$ of each agent a_i from graph g is given by:

$$u_i\left(g\right) = \sum_{s_j \in S} {b_j}^{t_{ij}} - \sum_{s_j \in N_g(a_i)} c_{ij}$$

where t_{ij} is the number of social groups in the shortest path between a_i and s_j (setting $t_{ij} = \infty$ if there is no path between a_i and s_j), $N_g(a_i)$ is the set of social groups of which a_i is a member, and $b_j \in (0, 1)$ captures the idea that the benefit that an agent derives from a social group is proportional to the proximity of the agent to the group.

It is necessary to clarify the condition of efficient graphs and stable graphs to analyze the network formation process. Firstly, we use the definition of strong efficiency in our analysis (Jackson and Wolinsky, 1996). The total value $v : g \to \mathbb{R}$ of a bipartite graph g is given by:

$$v\left(g\right) = \sum_{a_i \in A} u_i(g).$$

A bipartite graph g is said to be strongly efficient if $v(g) \geq v(g'), \forall g' \subseteq g^{m,n}$, where $g^{m,n}$ denotes the complete bipartite graph, i.e., the network in which each agent is a member of each social group.

Secondly, regarding the notion of stability, we employ the internal condition and the external condition developed in the open membership games (Selten, 1973; d'Aspermont et al., 1983). A bipartite graph between agents and social groups is stable if no member wants to leave out groups they participate in, and no outsider wants to join any group. Using these stable conditions, we assume that agents can form or sever links unilaterally. Hence, the bipartite graph g is stable if:

(i) for all $ij \in g$, $u_i(g) > u_i(g - ij)$; and

(ii) for all $ij \notin g$, $u_i(g) > u_i(g + ij)$.

3 The two social groups formation model

In this section, we focus on a society in which agents consider to participate in two social groups s_1 and s_2, in order to establish their social communication. Members of each group directly exchange information with each other. Since the information shared among members of a social group is the same, the benefits they receive are similar. By contrast, each member has to pay different cost to participate in a social group depending on their own individual conditions.

If there exists at least one agent participating in both groups, the information known by members of group s_1 could be shared with group s_2 and vice versa. It should be noted that the information exchange process between two groups is slower and less effective than among members of a given group. Therefore, the benefits from indirect communication between members of the two groups must be less than the benefit of directly being a member of a social group.

More precisely, each agent $a_i \in A$ pays a cost c_{ij} for being a member of a social group s_j for $j = 1, 2$, and receives a direct benefit $b_j \in (0, 1)$. If there is a path between s_1 and s_2, indirect benefits that each member of s_1 and s_2 receives are ${b_2}^2 < b_2$ and ${b_1}^2 < b_1$, respectively.

Considering the graph in which two social groups s_1 and s_2 are connected, we introduce the concept of **a star** which plays a central role in discussions about the stability and the efficiency of bipartite graph structures in the simple case of two social groups.

Definition 1. *The star a_i of graph g in the two social groups formation model is the sole agent who has an incentive to participate in both groups s_1 and s_2.*

In other words, being a member of both groups s_1 and s_2 brings the star a_i a positive utility: $u_i(g) = b_1 + b_2 - c_{i1} - c_{i2} > 0$. Let P_1 be the set of agents connected to s_1 and P_2 be the set of agents connected to s_2. Agent a_i is a star in g if $P_1 \cap P_2 = \{a_i\}$.

Figure 2 provides an example of a bipartite graph structure with a star a_i who participates in both groups s_1 and s_2. Note that the star a_i forms a unique bridge between s_1 and s_2, through two links $i1$ (between a_i and s_1) and $i2$ (between a_i and s_2). In other words, every path between any two

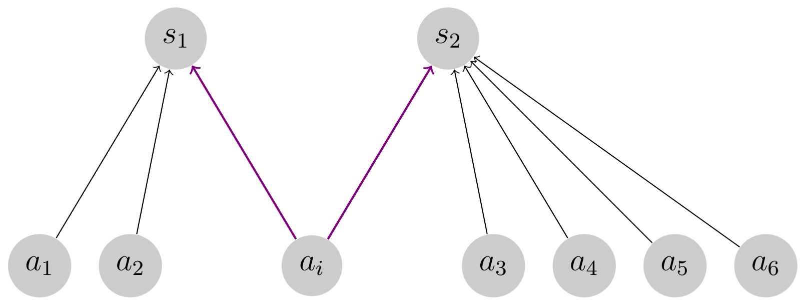

Figure 2: A bipartite graph structure with a star in the two social groups model

members of two groups s_1 and s_2 has to pass through the star a_i. A bipartite graph containing a star is acyclic.

Let $S = \{s_j, s_l\}, j, l = 1, 2, j \neq l$ and consider the option of agent a_i who is a member of the group s_l. Participation of a_i in the social group s_j brings her a direct utility $u_{ij} = b_j - c_{ij}$. Otherwise, if there exists a link between s_j and s_l, the indirect utility that a_i receives from s_j will be ${b_j}^2$. Therefore, agent a_i will participate in the group s_j if the direct benefit she gets from this membership is larger than the indirect benefit she could get thanks to the star, i.e. $c_{ij} \leq b_j - {b_j}^2$. Let $A_j \neq \emptyset$ be the set of agents for whom $c_{ij} < b_j - {b_j}^2$. Similarly, let $A_l \neq \emptyset$ be the set of agents for whom $c_{il} < b_l - {b_l}^2$.

Proposition 1. ***Efficiency***. *The unique strongly efficient graph in the two social groups formation model is*

(i) *the complete bipartite graph $g^{2,n}$ if $A_j \equiv A_l \equiv A$,*

(ii) *a connected acyclic bipartite graph if $A_l = A \backslash A_j$ and the bipartite graph contains a star,*

(iii) *a connected bipartite graph if $A_j \cup A_l = A$ and $A_j \cap A_l \neq \emptyset$,*

(iv) *the empty bipartite graph if $c_{ij} > b_j + {b_l}^2$ and $c_{il} > b_l + {b_j}^2$ hold for all agents.*

Proof. (i). $A_1 \equiv A_2 \equiv A$ means that ${b_j}^2 < b_j - c_{ij}$ and ${b_l}^2 < b_l - c_{il}$ hold for all agents. Considering the utility of agent a_i when she participates in both groups s_j and s_l: $u_i = b_j + b_l - c_j - c_l$. Hence, the overall value of the complete bipartite graph $g^{2,n}$ is given by:

$$v\left(g^{2,n}\right) = n(b_j + b_l) - \sum_{a_i \in A} (c_{ij} + c_{il})$$

Considering the bipartite graph in which agent a_i deletes her link with the social groups s_j, the overall value of the graph $(g^{2,n} - ij)$ is given by:

$$v\left(g^{2,n} - ij\right) = v\left(g^{2,n}\right) - b_j + c_{ij} + {b_j}^2$$

Since ${b_j}^2 < b_j - c_{ij}$ and ${b_l}^2 < b_l - c_{il}$ hold for all agents, we have $v(g^{2,n} - ij) < v(g^{2,n})$, $\forall \{ij\} \in L$. Finally, the complete bipartite graph $g^{2,n}$ is the unique strong efficient graph.

(ii). Let g be the bipartite graph in which all agents in A_j are connected to s_j, all agents in A_l are connected to s_l and no other direct link exists. Let the numbers of members in each group s_j and s_l be n_j and n_l, respectively. Hence, there are $n_j + n_l$ direct links in g. Since $A_j \neq \emptyset$ and $A_l = A \setminus A_j \neq \emptyset$, g is an acyclic bipartite graph encompassing every agent and every social group. The overall value of g is:

$$v(g) = n_j b_j + n_l b_l - \sum_{a_i \in A_j} c_{ij} - \sum_{a_i \in A_l} c_{il}.$$

Note that $v(g) > v(g')$ for all $g' \subset g$ since $A_j = \{a_i \in A \text{ s.t. } {b_j}^2 < b_j - c_{ij}\}$ and $A_l = \{a_i \in A \text{ s.t. } {b_l}^2 < b_l - c_{il}\}$. Considering the bipartite graph $g + ij$ in which one agent $a_i \in A_l$ creates a direct link to s_j to become a star. The overall value of $g + ij$ is:

$$v(g + ij) = v(g) + b_j + (n_l - 1)\,{b_j}^2 + n_j {b_l}^2 - c_{ij}$$

Since there is no link between two groups s_j and s_l in g, a_i participates in s_j if and only if $c_{ij} < b_j$. As a result, $v(g + ij) > v(g)$.

Note that $v(g + ij) > v(g' + ij)$ for all $g' \subset g$ since $v(g) > v(g')$ for all $g' \subset g$ and the indirect benefits induced by ij are strictly maximized in g. Hence a strongly efficient graph containing a star must be connected and acyclic.

(iii). Given that $A_j \cup A_l = A$, any agent belongs to either A_j or A_l or both. Moreover, given that $A_j \cap A_l \neq \emptyset$, there exists at least one agent belonging to both A_j and A_l. Hence, it follows that a strongly efficient graph must be connected (but not necessarily acyclic since the set $A_j \cap A_l$ may contain several agents).

(iv). Given that $c_{ij} > b_j + {b_l}^2$ and $c_{il} > b_l + {b_j}^2$ hold for all agents, the empty graph has higher value than any graph in which each agent is connected to at most one social group. Consider a bipartite graph g in which some agents are connected to s_j only, some agents are connected to s_l only, and one agent is connected to both s_j and s_l. The overall value of g is:

$$v(g) = p_j b_j + p_l b_l + (p_j - 1)\,{b_l}^2 + (p_l - 1)\,{b_j}^2 - \sum_{a_i \in P_j} c_{ij} - \sum_{a_i \in P_l} c_{il},$$

where P_j is the set of agents connected to s_j, $p_j = \#P_j$, and P_l is the set of agents connected to s_l, $p_l = \#P_l$, with:

$$v(g) < p_j \sum_{a_i \in P_j} \{b_j + {b_l}^2 - c_{ij}\} + p_l \sum_{a_i \in p_l} \{b_l + {b_j}^2 - c_{il}\} < 0.$$

To complete the proof, note that $v(g) > v(g')$ for all $g' \supset g$ since no new direct link added to graph g would induce indirect benefits. □

Proposition 2. ***Stability***. *The unique stable graph in the two social groups formation model is*

(i) *the complete bipartite graph $g^{2,n}$ if $A_j \equiv A_l \equiv A$,*

(ii) *a connected acyclic bipartite graph if $A_l = A \backslash A_j$ and the bipartite graph contains a star,*

(iii) *a disconnected bipartite graph encompassing every agent and every social group if $A_l = A \setminus A_j$ and the bipartite graph contains no star.*

Proof. (i). $A_1 \equiv A_2 \equiv A$ means that ${b_j}^2 < b_j - c_{ij}$ and ${b_l}^2 < b_l - c_{il}$ hold for all agents. Any agent who is not directly connected to s_1 and s_2 will improve her utility, and thus the total value, by forming both links.

(ii). Let g be the bipartite graph in which all agents in A_j are connected to s_j, all agents in A_l are connected to s_l. Since $A_j \neq \emptyset$ and $A_l = A \setminus A_j \neq \emptyset$, there is no member of group s_j is member of group s_l and vice versa. There are $n_j + n_l$ direct links in g and g is an acyclic bipartite graph encompassing every agent and every social group. The utility of each agent a_i from graph g is given by;

$$u_i(g) = \begin{cases} b_j - c_{ij} & \text{if } a_i \in A_j; \\ b_l - c_{il} & \text{if } a_i \in A_l. \end{cases}$$

Note that $u_i(g) > u_i(g - ij)$ for all $ij \in g \subseteq g^{2,n}$. Since there is no link between two groups s_j and s_l in g, a star a_i who receives a positive utility from being a member of both groups s_j and s_l will form links to connected both s_1 and s_2. The utility that the star achieved from the bipartite graph g is dominated by the utility that she obtains from the bipartite graph $g + ij$. Note that $g + ij$ is connected and contains a star, thus it is a connected acyclic bipartite graph.

To complete the proof, note that all other agents prefer being connected to only one social group since:

$$u_t(g + ij) = b_j + {b_l}^2 - c_{tj} > b_j + b_l - c_{tj} - c_{tl} = u_t(g + ij + tl)$$

for all $a_t \in A_j = A \setminus A_l$, $j, l = 1, 2$, $j \neq l$, $a_t \neq a_i$.

(iii). If there is no star, all agents $a_i \in A_j$ connect to only in the social group s_j since:

$$u_i(g) = b_j - c_{ij} > b_j + b_l - c_{ij} - c_{il} = u_i(g + il)$$

Similarly, all agents $a_i \in A_l$ only connect to the social group s_l. Therefore, in equilibrium, a disconnected bipartite graph that encompasses every agent and every social group will be stable. □

Corollary 1. *When comparing the efficiency and stability conditions presented above, the following remarks hold:we note that:*

(i) *If $A_j \equiv A_l \equiv A$, $j, l = 1, 2$, the complete bipartite graph $g^{2,n}$ is the unique strongly efficient and stable graph in equilibrium;*

(ii) *If $A_l = A \backslash A_j$, $j, l = 1, 2$, and the condition $b_j < c_{ij} < b_j + (n_l - 1) {b_j}^2 + n_j {b_l}^2$ is satisfied for one and only one agent $a_i \in A$, then a connected acyclic bipartite graph containing a star a_i is the unique strongly efficient graph but cannot be reached in equilibrium;*

(iii) *If $A_l = A \backslash A_j$, $j, l = 1, 2$, the condition $b_j < c_{ij} < b_j + (n_l - 1) {b_j}^2 + n_j {b_l}^2$ is satisfied for one and only one agent $a_i \in A$ and the bipartite graph contains no star, then a disconnected bipartite graph encompassing every agent and every social group is a unique stable graph in equilibrium but are not efficient.*

Proof. See Appendix 1. □

Obviously there is a common point between the star in the two social groups formation model and the central agent of the star in the Connection Model (Jackson and Wolinsky, 1996). The two concepts describe a unique agent who directly receive information from all of other agents. Thanks to its memberships in both groups, the star in our model directly exchanges information with all other agents as the central agent in the Connection Model (Jackson and Wolinsky, 1996). However, it should be noted that Jackson and Wolinsky (1996) defined his star as a graph in which all agents are connected with the central agent and there is no other link among other agents, thus, these agents cannot directly communicate to each other. By contrast, members of the same group of the two social groups formation model create direct link to each other including the star.

4 The three social groups formation model

In this section, we focus on the case of three social groups. It will be established later that this case is a representative example of the m social group formation models ($\forall m > 2$). We show that there exists situations in which, not only one agent, but several ones, play vital role in linking groups and facilitate the information exchange progress among agents in a society. We start our discussion by introducing two new notions - **the grand star** and **mini stars**. They are natural extensions of the star defined in the two social groups formation model.

Definition 2. *The grand star a_i of graph g in the m social groups formation model ($\forall m > 2$) is the sole agent who has an incentive to participate in all groups $s_j \in S$, while other agents do not want to belong to more than one group.*

In other words, being a member of all groups $s_j \in S$ bring the star a_i a positive utility: $u_i(g) = \sum_{s_j \in S} b_j - \sum_{s_j \in S} c_{ij} > 0$. For each $a_t \neq a_i$, there exits no more than one $s_j \in S$ such that $u_t(g) = b_j - c_{tj} > 0$. Let P_j is the set of agents connected to $s_j \in S$. Agent a_i is a grand star in g if there exists only one a_i such that $\bigcap_{s_j \in S} P_j = \{a_i\}$ and for every pair of s_j and $s_k \in S$, $j \neq k$, $P_j \cap P_k = a_i$.

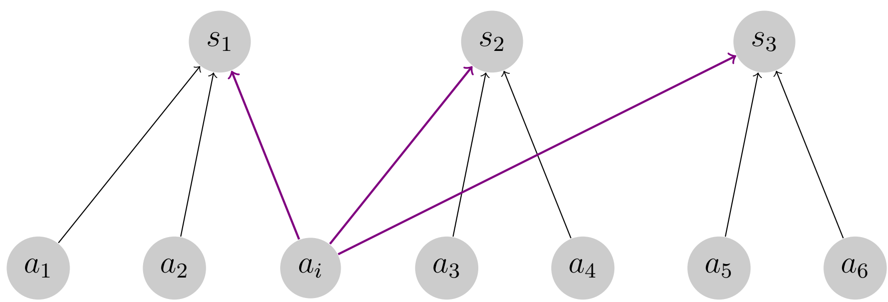

Figure 3: A bipartite graph structure with a grand star in the three social groups model

Figure 3 illustrates an example of a bipartite graph structure with a grand star a_i who belongs to all groups $s_j \in S$, $j = 1, 2, 3$. Note that the grand star a_i forms a unique bridge between any two groups s_j and $s_l \in S$ through two links ij (between a_i and s_j) and il (between a_i and s_l). In other words, every path between any two members of any two groups has to pass through the grand star a_i. A bipartite graph containing a grand star is acyclic.

Definition 3. *A set of mini stars $\{a_i\}$ of graph g in the m social groups formation model includes agents who have incentives to participate in at least* *two groups in S to ensure that all groups $s_j \in S$ are connected, but any two mini stars cannot simultaneously be members in two (or more) same groups.*

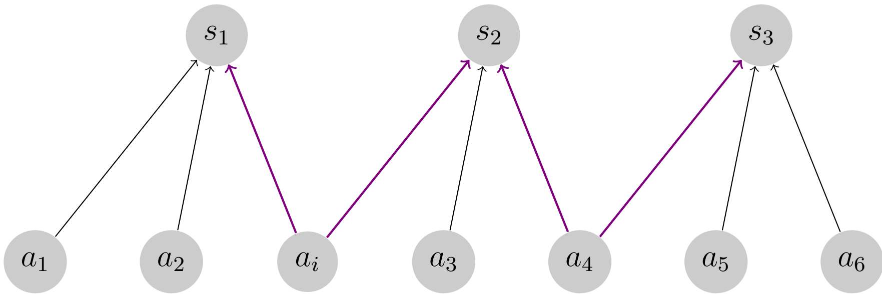

Figure 4: A bipartite graph structure with two mini stars in the three social groups model

Let M be the set of mini stars and μ is the number of mini stars in g. The following properties hold:

(i) a mini star $a_i \in M$ forms a unique bridge between any two groups s_j and $s_l \in S$ of which a_i is a member, through two links ij (between a_i and s_j) and il (between a_i and s_l). In other words, every path between any two members of any two groups has to pass through a mini star $a_i \in M$.

(ii) A bipartite graph containing mini stars is acyclic.

(iii) $\mu \leq m - 1$

Now, let's consider a society in which there exist three social groups. Let $S = \{s_j, s_l, s_k\}, j, l, k = 1, 2, 3, j \neq l \neq k$. Let $A_j \neq \emptyset$ be the set of agents for whom $c_{ij} < b_j - {b_j}^2$; $A_l \neq \emptyset$ be the set of agents for whom $c_{il} < b_l - {b_l}^2$; and let $A_k \neq \emptyset$ be the set of agents for whom $c_{ik} < b_k - {b_k}^2$.

Proposition 3. ***Efficiency***. *The unique strongly efficient graph in the three social groups formation model is:*

(i) *the complete bipartite graph $g^{3,n}$ if $A_j \equiv A_l \equiv A_k \equiv A$,*

(ii) *a connected acyclic bipartite graph if $A_l = A \setminus A_j \setminus A_k$ and the bipartite graph contains a grand star or mini stars,*

(iii) *a connected bipartite graph if $A_j \cup A_l \cup A_k = A$ and for every pair of s_j, $s_k \in S$: $A_j \cap A_k \neq \emptyset$,*

(iv) *the empty bipartite graph if* $c_{ij} > b_j + {b_l}^2 + {b_k}^2$, $c_{il} > b_l + {b_j}^2 + {b_k}^2$ *and* $c_{ik} > b_k + {b_j}^2 + {b_l}^2$ *hold for all agents.*

Proof. See Appendix 2. □

Corollary 2. *We define the bridge* L_g *as the set of links between any two groups if there is a grand star in* g *and* $n_{Lg} = \#L_g$. *Similarly, the bridge* L_m *is the set of links between any two groups if there is a set of mini stars in* g *and* $n_{Lm} = \#L_m$. *Thanks to the direct connection with all social groups* $s_j \in S$, *we always have* $n_{Lm} \leq n_{Lg}$.

Proof. See Appendix 3. □

Proposition 4. *Stability*. *The unique stable graph in the three social groups formation model is:*

(i) *the complete bipartite graph* $g^{3,n}$ *if* $A_j \equiv A_l \equiv A_k \equiv A$,

(ii) *a connected acyclic bipartite graph if* $A_l = A \setminus A_j \setminus A_k$ *and the bipartite graph contains a grand star or mini stars,*

(iii) *a disconnected bipartite graph encompassing every agent and every social group if* $A_j \cup A_l \cup A_k = A$, $A_j \cap A_l \cap A_j = \emptyset$ *and the bipartite graph contains neither a grand star nor mini stars.*

Proof. See Appendix 4. □

Corollary 3. *Comparing the efficiency and stability conditions presented above, we note that:*

(i) *If* $A_j \equiv A_l \equiv A_k \equiv A$, $j, k, l = 1, 2, 3$, *the complete bipartite graph* $g^{3,n}$ *is the unique strongly efficient and stable graph in equilibrium;*

(ii) *If* $A_l = A \setminus A_j \setminus A_k$, $j, k, l = 1, 2, 3$, *and the condition* $b_j < c_{ij} < b_j + (n_l - 1)\,{b_j}^2 + n_j {b_l}^2$ *is satisfied for at least one* $a_i \in A$, *a connected acyclic bipartite graph containing a grand star or a set of mini stars is the unique strongly efficient graph but cannot be reached in equilibrium;*

(iii) *If* $A_l = A \setminus A_j \setminus A_k$, $j, k, l = 1, 2, 3$, *the condition* $b_j < c_{ij} < b_j + (n_l - 1)\,{b_j}^2 + n_j {b_l}^2$ *is satisfied for at least one* $a_i \in A$, *and the bipartite graph contains no grand star, neither a set of mini stars, then a disconnected bipartite graph encompassing every agent and every social group is the unique stable graph in equilibrium but is not efficient.*

Proof. See Appendix 4. □

5 Discussion on the endogenous formation of m social groups

From propositions in sections 3 and 4, we can see some similarities between the conditions for efficiency and stability in the cases of two and three social groups formation models. However, because of the simplicity of the two social groups formation model, we can only see the existence of a unique star that connects both groups s_j and s_l but not the distinctive roles of a grand star and mini stars like in the three social groups formation model. Propositions 3 and 4, in fact, apply not only to the case of three social groups formation but also to the m social groups formation model.

When the cost of being member in social groups are sufficiently small, agents prefer to participate in all social groups to directly receive information. However, the number of social groups increases along with the development of modern society, and even if the membership fees of social groups are small enough, the time spends by people in these social groups does matter. Nguyen et al. (2016) showed that although the participation in social groups bring more happiness to people, it reduces their working time, and therefore, negatively affects their income. That is, the cost of being a member of all social groups is so large that people will have to choose in which community they should connect in order to maximize their utility. Such a decision will depend on, and define, the people priorities among the groups, as the choice of specific retailers, sport groups. In those situations, the grand star or mini stars play a vital role to ensure that the whole society is well connected and the structure of group memberships is efficient.

While the grand star is the solely central agent who participates in all social groups, and thus, connect all other agents, mini stars are a set of agents who simultaneously participate in several groups (but not all of them) to ensure that all groups and agents in the society are connected. Although our propositions show that the role of the grand star and mini stars are similar, it should be noticed that in reality the quality of information exchange process could be reduced if only one people know all the information of the whole society. Moreover, consider the time constraint as a part of membership cost, the grand star is unrealistic in sophisticated society with a vast number of social groups.

Mini stars, by contrast, is a set of agents who participate in more than one groups and help the diffusion of information through the whole society. While a bipartite graph containing mini stars is an ideal structure to ensure that the information exchange process is stable and efficient, as we shown, under certain conditions, a graph structure in which there exist several common members between any two social groups is also efficient.

Comparing with Jackson and Wolinsky (1996), our paper focuses on the bipartite connection between agents and social groups they are participate in rather than direct unimodal links among agents in a society. The stability and the efficiency of bipartite graphs are characterized through not only a central agent (the grand star) but also a set of semi-central agents (mini stars). We also show that, although there exist bipartite graph structures which are simultaneously stable and efficient, there exists situations in which a stable bipartite graph is not efficient and vice versa.

Appendix 1

Proof of Corollary 1.

Proof. (i). It could be derived from sections (i) of propositions 3 and 4.

(ii). Note that the overall value of $g+ij$ in the proof (ii) of the proposition 1 is:

$$v\left(g+ij\right) = v\left(g\right) + b_j + \left(n_l - 1\right) {b_j}^2 + n_j {b_l}^2 - c_{ij}$$

Therefore, $v(g+ij) > v(g)$ if $c_{ij} < b_j + (n_l - 1){b_j}^2 + n_j {b_l}^2$. When a connected acyclic bipartite graph containing a star is the unique stable graph in the two social group formation model, it is certainly that $c_{ij} < b_j < b_j + (n_l - 1){b_j}^2 + n_j {b_l}^2$. As a result, the graph is also the unique strongly efficient.

However, considering the case in which c_{ij} satisfies the following inequality:

$$b_j < c_{ij} < b_j + \left(n_l - 1\right) {b_j}^2 + n_j {b_l}^2$$

In equilibrium, a connected acyclic bipartite graph containing a star is still strongly efficient but not stable anymore.

(iii). Consider the graph g in which all agents $a_i \in A_j$ connect to s_j, all agents $a_i \in A_l$ connect to s_l and there is no other links exist. Since $A_l = A \setminus A_j$, from (iii) of proposition 2, g is the unique stable graph in the two social groups formation model.

Consider the case in which c_{ij} satisfies the following inequality:

$$b_j < c_{ij} < b_j + \left(n_l - 1\right) {b_j}^2 + n_j {b_l}^2$$

If there is no star, all agents $a_i \in A_j$ have no incentive to connect to s_l and vice versa. However, as showed in (ii) of the Comment 1, since $c_{ij} < b_j < b_j + (n_l - 1){b_j}^2 + n_j {b_l}^2$, a connected acyclic bipartite graph containing a star certainly dominates a disconnected bipartite graph encompassing every agent and every social group to become the strongly efficient graph.

□

Appendix 2

Proof of Proposiition 3.

Proof. (i). $A_j \equiv A_l \equiv A_k \equiv A$ means that ${b_j}^2 < b_j - c_{ij}$ holds for all agents $a_i \in A$ and all groups $s_j \in S$. Considering the utility of each agent when she participates in all social groups: $u_i = b_j + b_l + b_k - c_j - c_l - c_k$. Hence, the overall value of the complete bipartite graph $g^{3,n}$ is given by:

$$v\left(g^{3,n}\right) = n(b_j + b_k + b_l) - \sum_{a_i \in A} (c_{ij} + c_{ik} + c_{il})$$

Considering the bipartite graph in which an agent a_i deletes her link with the social groups s_j, the overall value of the graph $(g^{3,n} - ij)$ is given by:

$$v\left(g^{3,n} - ij\right) = v\left(g^{3,n}\right) - b_j + c_{ij} + {b_j}^2$$

Since ${b_j}^2 < b_j - c_{ij}$ holds for all agents $a_i \in A$ and all groups $s_j \in S$, we have $v(g^{3,n} - ij) < v(g^{3,n})$ $\forall \{ij\} \in L$. Finally, the complete bipartite graph $g^{3,n}$ is the unique strong efficient graph.

(ii). Let g be the bipartite graph in which all agents in A_j are connected to s_j, $\forall s_j \in S$, and no other direct link exists. Let the numbers of members in each group s_j, s_k and s_l are n_j, n_k and n_l, respectively. Hence, there are $n_j + n_k + n_l$ direct links in g. Since $A_l = A \setminus A_j \setminus A_k$, g is an acyclic bipartite graph encompassing every agent and every social group. The overall value of g is:

$$v(g) = n_j b_j + n_k b_k + n_l b_l - \sum_{a_j \in A_j} c_{ij} - \sum_{a_k \in A_k} c_{ik} - \sum_{a_l \in A_l} c_{il}$$

Note that $v(g) > v(g')$ for all $g' \subset g$ since for each agent $a_i \in A_j$, $c_{ij} < b_j - {b_j}^2$ with $\forall s_j \in S$.

(The overall value of the bipartite graph contains a grand star). Without loss of generality, considerer the bipartite graph $(g + ij + ik)$ in which one agent $a_i \in A_l$ creates links to s_j and s_k and becomes the grand star. The overall value of $(g + ij + ik)$ is given by:

$$v(g + ij + ik) = v(g) + \Delta_{lj} + \Delta_{lk} + \Delta jk - c_{ij} - c_{il}$$

in which:

$$\Delta_{lj} = [b_j + (n_l - 1){b_j}^2 + n_j {b_l}^2]$$

$$\Delta_{lk} = [b_k + (n_l - 1){b_k}^2 + n_k {b_l}^2]$$

$$\Delta jk = n_j {b_k}^2 + n_k {b_j}^2$$

Δ_{lj} is the benefit that members of groups s_j and s_l receive when s_j and s_l are connected by a_i. Similarly, Δ_{lk} is the benefit that members of groups s_k and s_l receive when s_k and s_l are connected by a_i. Finally, Δjk is added social welfare that agents in groups s_j and s_k obtain when the graph turns to be connected thanks to the grand star a_i.

With the appearance of the star a_i, the bipartite graph $v(g + ij + ik)$ is connected and acyclic. Note that there is no links among s_j, s_k, s_l in g, $a_i \in A_l$ participates in s_j and s_k if and only if $c_{ij} < b_j$ and $c_{ik} < b_k$. Therefore, we have:

(1) $v(g + ij + ik) > v(g' + ij + ik)\ \forall g' \subset g$;

(2) $v(g + ij + ik) > v(g + ij) > v(g' + ij)$ and $v(g + ij + ik) > v(g + ik) > v(g' + ik)\ \forall g' \subset g$.

Since the total value of the whole graph $(g+ij+ik)$ induced by the memberships of the grand star a_i strictly dominates the value of other possible graphs, it will be the unique strongly efficient graph.

(The overall value of the bipartite graph contains mini stars). Similar to the case of the bipartite graph containing a grand star. It is easy to show that the bipartite graph containing mini star is also acyclic and connected, meanwhile it produces the largest overall utility among possible graphs. The overall value of the bipartite graph containing two mini stars in the three social groups formation model is given by:

$$v\left(g + ij + tk\right) = v\left(g\right) + \Delta_{lj} + \Delta_{jk} + \Delta_{lk} - c_{ij} - c_{tk}$$

if an agent $a_i \in A_l$ has enough incentive to join in group s_j and an agent $a_t \in A_j$ has enough incentive to join in group s_k.

(iii). Given that $A_j \cup A_k \cup A_l = A$, any agent belongs to at least one social group. Moreover, given that for every pair of s_j, $s_k \in S$: $A_j \cap A_k \neq \emptyset$, there exist at least one agent between any two social groups. Hence, it follows from (ii) that a strongly efficient graph must be connected (but not necessarily acyclic since the set $A_j \cap A_k \cap A_l$ may contain several agents).

(iv). From collorary 3, we know that the grand star creates the shortest link between any two social groups. If there is a grand star $a_i \in A$ in a bipartite graph g, the utility of an agent $a_t \neq a_i \in A_l$ is $u_t(g) = b_l + {b_j}^2 + {b_k}^2 - c_{ij}$, $\forall s_j \in S$. The overall value of the bipartite graph g containing a grand star is given by:

$$v(g) = p_j b_j + p_l b_l + p_k b_k + (p_l - 1)(b_j^2 + b_k^2) + p_k(b_l^2 + b_j^2) + p_j(b_l^2 + b_k^2)$$
$$- \sum_{a_i \in P_j} c_{ij} - \sum_{a_i \in P_k} c_{ik} - \sum_{a_i \in P_l} c_{il}$$

where P_j is the set of agents connected to s_j, $p_j = \#P_j$, P_l is the set of agents connected to s_l, $p_l = \#P_l$, and P_k is the set of agents connected to s_k, $p_k = \#P_k$. Observe that:

$$v(g) < p_j \sum_{a_i \in P_j} \{b_j + {b_k}^2 + {b_l}^2 - c_{ij}\} + p_k \sum_{a_i \in P_k} \{b_k + {b_j}^2 + {b_l}^2 - c_{ik}\} +$$

$$p_l \sum_{a_i \in P_l} \{b_l + {b_j}^2 + {b_k}^2 - c_{il}\} < 0.$$

In the bipartite graph containing mini stars, the utilities of other agents are reduced since mini stars may lengthen the distance between two any social groups. Similar to the case of the bipartite graph containing a grand star, we can easily show that the value of bipartite graphs containing mini stars satisfies with the above inequality.

(iii) To complete the proof, note that $v(g) > v(g')$ for all $g' \supset g$ since no new direct links added to graph g would induce indirect benefits. □

Appendix 3

Proof of Colorrary 2.

Proof. A bridge in g between two social groups, say s_j and s_l, is a set of links $\{j_1 i_1, i_1 j_2, j_2 i_2, \ldots, j_{p-1} i_p, i_p j_p\}$, in which $j_1 = j$ and $j_p = l$ that connect s_j

Consider the bipartite graph g that includes a grand star a_i, since a_i is a member of all three social groups s_j, s_l and s_k, bridges between any two groups in g simply include two links. For example, the bridges between s_j and s_l are: □

Appendix 4

Proof of Proposiition 4.

Proof. (i). $A_j \equiv A_l \equiv A_k \equiv A$ means that ${b_j}^2 < b_j - c_{ij}$ holds for all agents $a_i \in A$ and all groups $s_j \in S$. Any agent who is not directly connected to s_j benefits from forming links. Finally, all agents simultaneously participate in all social group to achieve the highest benefit, therefore, the complete bipartite graph is stable.

(ii). Let g be the bipartite graph in which all agents in A_j are connected to s_j $\forall s_j \in S$ and no other direct links exist. Hence there are $n_1 + n_2 + n_3$

direct links in g, and g is an acyclic bipartite graph encompassing every agent and every social group. The utility of each agent a_i from graph g is given by:

$$u_i(g) = \begin{cases} b_j - c_{ij} & \text{if } a_i \in A_j; \\ b_k - c_{ik} & \text{if } a_i \in A_k; \\ b_l - c_{il} & \text{if } a_i \in A_l. \end{cases}$$

Note that $u_i(g) > u_i(g - ij)$ for all $ij \in g \subseteq g^{3,n}$. Since there is no link between two any groups in g, the grand star who receives a positive utility from being member of all three groups will form links to connect all groups $s_j \in S$ to maximize her utility. Since all $s_j \in S$ is connected now, the bipartite graph g turns to be a connected bipartite graph. The new connected bipartite graph contains a grand star, thus it is a connected acyclic bipartite graph.

Similarly, consider another case in which there exit two agents who receive positive utilities by forming a link with one more group and become two mini-stars, these mini stars also create new links to maximize their utilities. As a results, all the social groups $s_j \in S$ as well as the bipartite graph are connected. And since the connected bipartite graph contains two mini stars, it is acyclic.

To complete the proof, note that all other agents who are not mini stars prefer being connected to only one social group following the definition of the mini stars.

(iii). Let g be the bipartite graph in which all agents in A_j are connected to s_j $\forall s_j \in S$ and no other direct links exist. g is acyclic bipartite graph encompassing every agent and every social group. Since $A_j \cap A_l \cap A_j = \emptyset$ and there is neither the grand star, nor mini stars in g, there is no link connecting all social groups $s_j \in S$. Therefore, g is a disconnected bipartite graph. In equilibrium, g is the unique stable graph since no agent $a_i \in A$ has any incentive to form links connected all social groups, and the whole graph as well. □

Appendix 4

Proof of Proposiition 4.

Proof. The similar explanation as in Appendix 1 is applied to prove the Proposition 4. □

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