Buyer-Optimal Robust Information Structures*

Stefan Terstiege† Cédric Wasser‡

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Abstract

We study buyer-optimal information structures under monopoly pricing. The information structure determines how well the buyer learns his valuation and affects, via the induced distribution of posterior valuations, the price charged by the seller. Motivated by the regulation of product information, we assume that the seller can disclose more if the learning is imperfect. Robust information structures prevent such disclosure, which is a constraint in the design problem. Our main result identifies a two-parameter class of information structures that implements every implementable buyer payoff. An upper bound on the buyer payoff where the social surplus is maximized and the seller obtains just her perfect-information payoff is attainable with some, but not all priors. Generally, optimal information structures may result in an inefficient allocation.

Keywords: information design, monopoly, regulation.

JEL-Classification: D42, D82, D83, L51.

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†Maastricht University, Department of Economics (AE1), P.O. Box 616, 6200 MD Maastricht, Netherlands. Email: s.terstiege@maastrichtuniversity.nl.

‡University of Bonn, Department of Economics, Lennéstr. 37, 53113 Bonn, Germany.
Email: cedric.wasser@uni-bonn.de.

1 Introduction

Before making a purchase decision, consumers typically try to assess how well the product under consideration matches their preferences, using various sources of information. Examples include technical specifications or a list of ingredients published by the seller, advertising, (online) reviews, product samples, and testing the product during a trial period. Whereas sellers often have considerable control over such information, its disclosure is regulated in many countries, with the aim of promoting consumer welfare. The European Union, for example, has passed regulation ranging from food information over insurance mediation to the content of financial security prospectuses. It has also introduced a mandatory period of 14 days during which consumers can withdraw from a sales contract concluded via the Internet.[1] Effectively, this period amounts to a trial period during which consumers can learn better to what extent the product fits their preferences.

Sellers are usually free to provide more information than the regulator requires. A trial period, for instance, can be extended beyond the obligatory number of days.[2] When setting minimum disclosure requirements, the regulator must therefore take into account how the requirements affect sellers' incentives to disclose more. More information is not necessarily advantageous for buyers: it allows better purchasing decisions, but if the information creates more dispersion in the buyers' willingness to pay, sellers may raise prices. Hence, what are buyer-optimal minimum disclosure requirements when the seller can disclose more? This is the question we address in this paper.

We take an information-design approach and study buyer-optimal information structures under monopoly pricing. In our model, the seller has a single object for sale, which she values at zero, and she faces one potential buyer. An information structure consists of a set of signals and probability distributions over signals conditional on the buyer's

[1] See, respectively, Regulation (EU) No 1169/2011, Directive 2002/92/EC, Regulation (EU) 2017/1129, and Directive 2011/83/EU.

[2] For example, in the European Union, the Apple online store accepts returns within the obligatory 14 days, whereas Amazon extended this period to 30 days, Zalando, an online fashion retailer, to 100 days, and IKEA to a full year.

valuation, that valuation being unknown to the buyer and the seller. At the outset, the buyer chooses an information structure. Afterwards, the seller sets a price and decides about releasing additional information. Specifically, she can extend the information structure by adding a signal component. At the end, the buyer privately observes the signal of the (possibly extended) information structure, updates to a posterior valuation, and decides whether or not to buy. Since any additional signal component can be incorporated at the outset, we restrict attention to information structures under which the seller has no incentive to disclose more. We call such information structures robust.

Our main result identifies a two-parameter class of information structures with the property that for every buyer payoff that can be implemented by *some* information structure, there exists an information structure in this class that implements this payoff. The information structures are characterized as follows. The two parameters determine an interval of valuations. All valuations outside this interval are disclosed perfectly. All valuations inside it are pooled, pairwisely and such that the posterior valuation is always the same. In particular, the pooling proceeds in a deterministic, negative assortative fashion: high valuations are pooled with low ones according to a specific decreasing matching function.

In the derivation of this result, we exploit a connection to matching, or optimal transport. We consider the problem of inducing a given buyer payoff while minimizing the seller's gain from disclosing more. We confine this problem to information structures that pool only the valuations inside some interval, pairwisely and such that the posterior valuation is always the same. Here, the pooling might still be stochastic. The key step is to establish an equivalence between such information structures and a certain class of all bivariate distributions with given marginals. Working with the bivariate distributions, we get an optimal-transport problem. This problem has a supermodular objective function, which implies that pooling in a deterministic, negative assortative fashion is optimal.

The main result narrows the search for buyer-optimal information structures down to the two parameters of the negative assortative information structures. A natural upper bound for the buyer payoff is characterized by trade with probability one, maximizing the social surplus, and the seller getting just her perfect-information payoff, which she can always secure by disclosing perfect information. For some priors (e.g., the uniform distribution) this upper bound can be attained, whereas for other priors it cannot. In general, the robustness constraint is not sufficiently tractable to obtain an analytical solution for the buyer-optimal values of the two parameters. The problem is, however, well suited for simulation. By means of numerical examples, we demonstrate that optimal information structures may result in the seller getting a strictly higher payoff than under perfect information and, at the same time, in a probability of trade strictly less than one and thus an inefficient allocation. Yet negative assortative information structures are constrained-efficient: for any given buyer payoff, they induce the highest possible corresponding seller payoff.

Our analysis contributes to the literature on information design (e.g., Kamenica and Gentzkow, 2011; Bergemann, Brooks, and Morris, 2015; Li and Shi, 2017). The most closely related paper is the one by Roesler and Szentes (2017), who also study buyer-optimal information structures under monopoly pricing but without disclosure by the seller. Their results provide a benchmark for evaluating the relevance of our robustness constraint. The constraint always binds: unconstrained optimal information structures yield the seller even less than her perfect-information payoff. Like us, Roesler and Szentes identify a class of information structures that implements every implementable buyer payoff. We show that their class need not contain an optimal information structure for our setting. In both settings, however, optimal information structures typically do not remove the buyer's uncertainty completely (see also Kessler, 1998).

Several recent papers also study information structures that pool types in a negative assortative fashion. Von Wangenheim (2017) shows that the same class of information structures as here implements every implementable combination of buyer and seller payoff in sequential screening.[3] The key difference is that the buyer eventually learns his valuation perfectly, whereas in our paper the seller endogenously decides how much information to add. Nikandrova and Pancs (2017) consider sequential two-bidder auctions with information acquisition. When recommending information acquisition to the

[3] We thank Jonas von Wangenheim for pointing us to this class.

second bidder, the auctioneer optimally pools high and low bids of the first bidder to mitigate incentive constraints. In an insurance model, Garcia, Teper, and Tsur (2018) show that optimal information structures pool risk types in a negative assortative fashion to minimize price dispersion.

Li and Norman (2017) study a general persuasion game where, as in our model, several players can disclose information sequentially (see Gentzkow and Kamenica, 2017, for simultaneous disclosure). Like here, attention can be restricted to equilibria in which subsequent players have no incentive to add information (see also Perez-Richet and Skreta, 2017). Concerning disclosure to a receiver who is privately informed, Kolotilin, Li, Mylovanov, and Zapechelnyuk (2017) establish a payoff equivalence between experiments and mechanisms that provide an experiment conditional on a report by the receiver (see also Guo and Shmaya, 2017). One interpretation of our model is that the buyer observes the signal from the original information structure before the seller decides about her disclosure. For the main result, we assume that the seller can directly add a correlated signal, but we also consider experiments.

While our focus is on buyer-optimal information structures, another strand of literature on information design studies *seller*-optimal information structures for various selling environments (see, e.g., Lewis and Sappington, 1994; Bergemann and Pesendorfer, 2007; Eső and Szentes, 2007; Board and Lu, 2017). The buyer in our model has no private information at the outset, and to maximize the social surplus, he should always get the object. Thus, the seller-optimal information structure would simply provide no information. A large and influential literature investigates the incentives of sellers to voluntarily disclose information that is objective (i.e., everybody can assess its relevance) and certifiable (i.e., the seller can prove the true state). According to the "unraveling" argument (Grossman and Hart, 1980; Milgrom, 1981), sellers automatically have an incentive to disclose such information. In our model, the argument does not apply: the relevance of the information to the buyer depends on the buyer's individual preferences, which the seller does not know (see also Koessler and Renault, 2012).

The rest of the paper is organized as follows. The next section presents the model. Section 3 illustrates our results for a uniform prior. In Section 4, we establish the main result on negative assortative information structures. Section 5 studies optimal information structures. In Section 6, we discuss a weaker robustness constraint, how the seller's ability do add information changes the design problem, and an alternative timing. Section 7 concludes. Most proofs are in the appendix.

2 Model

Payoffs and prior information. A seller has a single object to sell to a buyer. The buyer's valuation for the object is initially unknown to both parties. Both believe that it is drawn from the cumulative distribution function (CDF) F over $[0,1]$, which admits the strictly positive probability density function (PDF) f. The seller offers the object at a take-it-or-leave-it price p. If the buyer accepts the offer and has valuation v, then his payoff is $v-p$ and the seller's payoff is p. If the buyer rejects, payoffs are both zero.

Information structures. Before the buyer decides about the purchase, he receives information about his valuation. Specifically, he observes a signal from some information structure. An *information structure* is a combination $(S,(G_v))$ of a signal set S and CDFs G_v on S such that if the buyer has valuation v, then a signal $s \in S$ is drawn from G_v and privately observed by the buyer. A *perfect* information structure, for example, has CDFs G_v whose supports are disjoint across v, so that it reveals the valuation fully. More generally, an information structure is *partitional* if there exists a partition of the set of valuations $[0,1]$ such that if v', v'' belong to the same partition element, the CDFs G_v coincide, and across partition elements the supports of the CDFs are disjoint. The signal set S of an information structure is a subspace of some Euclidean space. Let $\bar{G}$ denote the unconditional CDF on S, that is,

$$\bar{G}(s) := \int_0^1 \int_{\{e \in S: e \leq s\}} dG_v(e) dF(v).$$

Actions and timing. There are three stages. First, the buyer (or a regulator) chooses an information structure $(S^a,(G_v^a))$. In the second stage, the seller observes $(S^a,(G_v^a))$ and sets a price p. Moreover, she decides about releasing additional information. Specifically, she can *extend* $(S^a, (G_v^a))$ to any information structure $(S, (G_v))$ with $S = S^a \times S^b$ for some S^b and $\int_{S^b} dG_v(\cdot, s^b) = G_v^a$.[4] In the third stage, the buyer observes the (possibly extended) information structure and the signal, updates his belief about his valuation, and decides whether or not to buy the object.

Posterior beliefs and posterior valuations. Upon observing signal $s \in S$ from information structure $(S, (G_v))$, the buyer updates his belief to a posterior distribution function F_s over valuations $v \in [0, 1]$. Formally, the posteriors are characterized by the condition that for all $V \in \mathcal{B}([0, 1])$ and all $M \in \mathcal{B}(S)$,

$$\int_M \int_V dF_s(v) d\bar{G}(s) = \int_V \int_M dG_v(s) dF(v), \tag{1}$$

where $\mathcal{B}(\cdot)$ denotes the respective Borel σ-algebra.[5] Hence, the posterior valuation upon observing s is $E[v|s] = \int_0^1 v dF_s(v)$, and so the information structure induces the CDF of posterior valuations

$$H(w) := \int_{\{s \in S: E[v|s] \leq w\}} d\bar{G}(s).$$

Note that under a perfect information structure, H coincides with the prior F.

We assume that the buyer purchases the object if and only if $E[v|s] \geq p$. Thus, given price p and a CDF of posterior valuations H, the (ex-ante) probability of trade is $1 - H(p) + \Delta(H, p)$, where $\Delta(H, p)$ denotes the probability of posterior valuation p under H.[6] An information structure *induces* price p, buyer payoff U, and seller payoff Π if $p \in \operatorname{argmax}_q [1 - H(q) + \Delta(H, q)]q$, $U = \int_p^1 (v - p) dH(v)$, and $\Pi = [1 - H(p) + \Delta(H, p)]p$.[7] In words, this means that without additional disclosure, the seller would be willing to charge price p and this price results in buyer payoff U and seller payoff Π. When the seller has no incentive to disclose more, we occasionally use the term *implement* instead of 'induce'.

[4] In Section 6.1, we consider the case that s^b must be conditionally independent of s^a.

[5] Thus, the posteriors F_s are the CDFs corresponding to a regular conditional distribution, which exists and is unique almost everywhere (see, e.g., Dudley, 2002, Thm. 10.2.2).

[6] Formally, $\Delta(H, p) := H(p) - \sup_{x<p} H(x)$, as in Roesler and Szentes (2017).

[7] Where no confusion results, we write "payoff" instead of "expected payoff", and similarly for surplus.

Our aim is to study the information structures that maximize the buyer payoff when the seller can disclose more. Let $(S^a, (G_v^a))$ be any information structure, and suppose it is optimal for the seller to extend $(S^a, (G_v^a))$ to $(S, (G_v))$. Then, $(S, (G_v))$ does not induce further disclosure. Accordingly, we confine the analysis to information structures under which the seller has no incentive to provide an extension (and we usually omit the superscripts a, b). We call such information structures *robust*.

3 Example: The Uniform Case

To illustrate our results, we construct here a buyer-optimal robust information structure for the special case where the prior is the uniform distribution (i.e., $F(v) = v$).

Because the seller can always extend to perfect information, she must get under any robust information structure at least her perfect-information payoff $\max_p(1-p)p = 1/4$. The maximum total surplus is $E[v] = 1/2$, which materializes if trade happens with probability one. Consequently, the buyer payoff, which is the difference between the total surplus and the seller payoff, can be at most $1/4$.

We will show that the following information structure attains this upper bound on the buyer payoff: If $v > 1/2$, display $s = v$ with probability one. Thus, the buyer learns his valuation perfectly. If $v \leq 1/2$, display $s = |v - 1/4|$ with probability one. Thus, for valuations $v \leq 1/2$ the buyer only learns the distance between his valuation and $1/4$, which leads to posterior valuation $1/4$. The distribution of posterior valuations is then

$$H(w) = \begin{cases} 0 & \text{if } w \in [0, \frac{1}{4}), \\ \frac{1}{2} & \text{if } w \in [\frac{1}{4}, \frac{1}{2}], \\ w & \text{if } w \in (\frac{1}{2}, 1]. \end{cases} \tag{2}$$

It is straightforward to verify that this information structure induces price $1/4$, that is, $1/4 \in \operatorname{argmax}_p[1 - H(p) + \Delta(H, p)]p$. Moreover, as trade happens at this price with probability one, the induced seller and buyer payoffs are both equal to $1/4$.

We now demonstrate that the above information structure is robust, that is, the seller cannot gain by extending it. To this end, we show that there is no combination of an extension and a price q that yields a seller payoff strictly greater than 1/4. Under any extension, prices below 1/4 or above 1/2 are strictly dominated by price 1/2, which just yields seller payoff 1/4. So take any price $q \in (1/4, 1/2)$ and suppose the seller chooses an extension that maximizes the probability of trade (and hence her payoff) at q. First note that for some valuations v, the signal s is already sufficiently informative such that no extension can change the buyer's decision: he always buys if $v \geq 1/2$ and he never buys if $v \in (1/2 - q,\, q)$. To maximize the probability of trade for the remaining valuations v, the seller can extend the information structure as follows: If $v \in [q, 1/2]$, display a signal *BUY* with probability one.[8] If $v \in [0, 1/2 - q]$, display *BUY* with probability

$$x(v) := \frac{\frac{1}{2} - v - q}{q - v}.$$

The buyer's posterior valuation upon observing $s \leq 1/4$ and *BUY* is exactly q:

$$E[v|s, BUY] = \frac{x(\frac{1}{4} - s) \cdot (\frac{1}{4} - s) + 1 \cdot (\frac{1}{4} + s)}{x(\frac{1}{4} - s) + 1} = q.$$

Consequently, for any s, the extension persuades the buyer to buy with probability one if $v \geq q$ and with the highest possible probability (i.e., $x(v)$ or 0) if $v < q$. The seller payoff with this extension is

$$\left(1 - q + \int_0^{\frac{1}{2}-q} x(v) dv\right) q < \left(1 - q + \int_0^{\frac{1}{2}-q} \frac{\frac{1}{2} - q}{q} dv\right) q = \frac{1}{4}.$$

Hence, the information structure is robust.

Note that there are many information structures that also induce the CDF of posterior valuations (2) but are not robust. For example, suppose all valuations above 1/2 are disclosed perfectly, whereas all valuations below are pooled into the same signal. In that case, the seller could provide the additional information of whether or not the valuation exceeds 1/4, charge price 3/8, and thereby obtain payoff $(3/4) \cdot (3/8) > 1/4$. Hence, for robustness the distribution of posterior *beliefs* matters, not just the distribution of posterior *valuations*.

[8] For convenience, we occasionally use terms such as "*BUY*" for particular signals.

4 Negative Assortative Information Structures

We now return to the general case, where the prior F is arbitrary, and show that the search for buyer-optimal information structures can be restricted to a two-parameter class of information structures, which we call "negative assortative". Every implementable combination of price and buyer payoff remains implementable when restricting to this class, along with the highest possible corresponding seller payoff. The optimal information structure for the uniform case in the preceding section belongs to this class.

We say that an information structure $(S, (G_v))$ is *p-pairwise* if for almost all signals s there exist valuations $v_L, v_H \in [0, 1]$, where $v_L \leq v_H$, such that the posterior belief F_s has support $\{v_L, v_H\}$ and

$$\text{either:} \quad v_L = v_H \tag{3}$$

$$\text{or:} \quad v_L < p < v_H \text{ and } E[v|s] = F_s(v_L)v_L + [1 - F_s(v_L)]v_H = p. \tag{4}$$

Thus, under a p-pairwise information structure the buyer deems at most two valuations possible upon observing the signal, and whenever he deems two valuations possible his posterior valuation is exactly p. The optimal information structure in Section 3 is p-pairwise (with $p = 1/4$) and partitional. In general, however, p-pairwise information structure need not be partitional.

Lemma 1. *For every robust information structure that induces price p, there exists a robust p-pairwise information structure that induces the same price, the same buyer payoff, and the same seller payoff.*

Invoking this lemma, we can restrict attention to p-pairwise information structures. The basic intuition is as follows. The price and the payoffs depend only on the CDF of posterior valuations. To deter disclosure by the seller, the CDF of posterior valuations should be implemented by an information structure that is already as informative as possible. Every information structure that pools more than two valuations into the same signal can be made more informative without changing posterior valuations. For example, suppose three valuations $v' < v'' < v'''$ are pooled into the same signal s, where

$E[v|s] \in (v'', v''')$. Then, one can instead pool v' with v''' and v'' with v''' into two distinct signals such that the posterior valuation is $E[v|s]$ after either one.

We adapt our notation to p-pairwise information structures. Notice that (3) and (4) uniquely pin down the posterior belief F_s. That is, if some signals induce posterior beliefs that have the same support, then these posterior beliefs coincide almost surely. All such signals can be merged. We therefore denote signals of a p-pairwise information structure directly by $s = (v_L, v_H)$, where $\{v_L, v_H\}$ is the support of F_s. For valuations $v < p$, we have almost surely either $v = v_L < v_H$ or $v = v_L = v_H$, that is, the support of G_v is contained in $\{v\} \times (\{v\} \cup [p, 1])$. Define for all $v < p$

$$G_v^H(v_H) := G_v(v, v_H).$$

Similarly, for valuations $v > p$ we have almost surely either $v = v_H > v_L$ or $v = v_H = v_L$, that is, the support of G_v is contained in $([0, p] \cup \{v\}) \times \{v\}$. Define for all $v > p$

$$G_v^L(v_L) := G_v(v_L, v).$$

Observe that for valuations $v < p$, the first component of the signal equals v and the second is drawn from G_v^H, and the buyer learns v perfectly with probability $G_v^H(p)$. Similarly, for valuations $v > p$, the first component is drawn from G_v^L and the second equals v, and the buyer learns perfectly with probability $1 - G_v^L(p)$.

Next, we turn to the seller's response against a p-pairwise information structure. For an arbitrary price q, we construct an extension that maximizes the probability of trade given that price, analogously to Section 3. Regardless of the extension, there will be trade with probability one after all signals $s = (v_L, v_H)$ with $v_L = v_H \geq q$. Moreover, there will be no trade after all s with $v_H < q$. Consider the following extension, performed for all signals s with $v_L < v_H \in [q, 1]$ (such signals exist only if $q > p$): If $v = v_H$, display a signal BUY with probability one. If $v = v_L$, display BUY with probability

$$x_q(v_L, v_H) := \frac{p - v_L}{v_H - p} \frac{v_H - q}{q - v_L}.$$

To see that this extension maximizes the probability of trade at price q, note that given (4) the posterior valuation upon observing s and BUY is exactly q:

$$E[v|s, BUY] = \frac{F_s(v_L) x_q(v_L, v_H) v_L + [1 - F_s(v_L)] v_H}{F_s(v_L) x_q(v_L, v_H) + 1 - F_s(v_L)} = q.$$

We call this extension *q-optimal*. The following lemma summarizes.

Lemma 2. *Under a q-optimal extension of a p-pairwise information structure, the probability of trade conditional on the true valuation v and the signal $s = (v_L, v_H)$ is*

- *one if $v = v_H \geq q$,*
- *$x_q(v_L, v_H)$ if $v = v_L < v_H \in [q, 1]$,*
- *and zero otherwise.*

We now consider the problem of designing a p-pairwise information structure that minimizes the seller's gain from any q-optimal extension while inducing a given buyer payoff. We will ultimately state this problem as an optimal transport problem, where the choice set is a set of all bivariate distribution functions with given marginals.

First, we establish an equivalence between p-pairwise information structures and certain bivariate distribution functions. We call a distribution function J on $[0, p] \times [p, 1]$ *p-pairwise* if for a function $\alpha\colon [0, 1] \to [0, 1]$ such that

$$\int_0^p \alpha(v)(p - v)dF(v) = \int_p^1 \alpha(v)(v - p)dF(v) =: c,$$

the marginals of J are

$$\begin{aligned} J^L(v_L) := J(v_L, 1) &= \frac{1}{c} \int_0^{v_L} \alpha(v)(p - v)dF(v), \\ J^H(v_H) := J(p, v_H) &= \frac{1}{c} \int_p^{v_H} \alpha(v)(v - p)dF(v). \end{aligned}$$

A p-pairwise information structure $(S, (G_v))$ and a p-pairwise distribution function J are *equivalent* if

$$J(v_L, v_H) = \frac{1}{c} \int_0^{v_L} \int_p^{v_H} dG_v^H(u)(p - v)dF(v) \quad \text{and} \quad \alpha(v) = \begin{cases} 1 - G_v^H(p) & \text{for } v < p, \\ G_v^L(p) & \text{for } v > p. \end{cases}$$

Lemma 3. *For every p-pairwise information structure, there exists an equivalent p-pairwise distribution function J and vice versa.*

Under a p-pairwise information structure, each valuation v is pooled into posterior valuation p with some probability $\alpha(v)$ and is perfectly disclosed with probability $1 - \alpha(v)$. To see what J measures, suppose that the seller charges price p under a p-pairwise information structure. If the buyer updates to posterior valuation p, and thus buys, he makes a loss whenever his true valuation is smaller than p. The marginal J^L expresses how these ex-post losses are distributed over the valuations $v \in [0, p]$. Similarly, the marginal J^H expresses how the profits that the buyer makes if the posterior valuation equals p while the true valuation is greater than p are distributed over $[p, 1]$. The bivariate distribution function J, finally, measures for each signal $s = (v_L, v_H)$ with posterior valuation p the loss and profit, respectively, that v_L and v_H contribute when pooled into that signal.

For our purposes, a p-pairwise information structure and the equivalent p-pairwise distribution function J are interchangeable.[9] Consider a p-pairwise distribution function J that induces price p. Observe that the induced buyer and seller payoff are

$$U = \int_p^1 (v - p) dF(v) - c, \tag{5}$$

$$\Pi = \left[1 - F(p) + \int_0^p \alpha(v) dF(v) \right] p. \tag{6}$$

We use J to quantify the probability of trade under a q-optimal extension. Define

$$\phi_q(v_L, v_H) := \max \left\{ \frac{(v_H - q)c}{(v_H - p)(q - v_L)}, 0 \right\}.$$

Then, according to Lemma 2, the probability of trade given price q is

$$\begin{aligned} &\int_0^p \int_p^1 \max\{x_q(v_L, v_H), 0\} dG^H_{v_L}(v_H) dF(v_L) + 1 - F(q) \\ =& \int_0^p \int_p^1 \phi_q(v_L, v_H) \frac{1}{c}(p - v_L) dG^H_{v_L}(v_H) dF(v_L) + 1 - F(q) \\ =& \int_S \phi_q(v_L, v_H) dJ(v_L, v_H) + 1 - F(q). \end{aligned}$$

Thus, using J the probability of trade under a q-optimal extension can be expressed as an expectation of the function ϕ_q. Importantly, this function is supermodular.

[9] An equivalent p-pairwise information structure is unique almost everywhere.

It turns out that we can concentrate on p-pairwise distribution functions J under which each valuation is either always or never pooled into posterior valuation p (i.e., $\alpha(v) \in \{0, 1\}$ for all v) and under which those valuations that are pooled constitute and interval. Observe that if the interval $[\underline{v}, \overline{v}]$ is pooled into p, then $\int_{\underline{v}}^{\overline{v}} (v - p) dF(v) = 0$, and so $\overline{v}$ is uniquely determined by p and $\underline{v}$. A p-pairwise distribution function J will be called $(p, \underline{v})$-*pairwise* if for the corresponding $\overline{v}$,

$$\alpha(v) = \begin{cases} 1 & \text{for } v \in [\underline{v}, \overline{v}], \\ 0 & \text{for } v \notin [\underline{v}, \overline{v}]. \end{cases}$$

Lemma 4. *For every robust p-pairwise distribution function that induces price p, there exists a robust* $(p, \underline{v})$*-pairwise distribution function that induces the same price, the same buyer payoff and a weakly higher seller payoff.*

Now, fix $p \in [0, 1]$ and $\underline{v} \leq p$ such that $(p, \underline{v})$-pairwise distribution functions induce price p. This also fixes the buyer and the seller payoff. Informally, it remains to pool the valuations $v_L \in [\underline{v}, p)$ pairwisely with the valuations $v_H \in (p, \overline{v}]$, possibly in a stochastic way, such that the posterior valuation is always p. Consider the problem of choosing a $(p, \underline{v})$-pairwise distribution function J that is "as robust" as possible. Specifically, choose J to minimize the probability of trade under any q-optimal extension:

$$\begin{aligned} \min_{J} \quad & \int_S \phi_q(v_L, v_H) dJ(v_L, v_H) \\ s.t. \quad & J^L(v_L) = \frac{1}{c} \int_{\underline{v}}^{v_L} (p - v) dF(v), \\ & J^H(v_H) = \frac{1}{c} \int_{p}^{v_H} (v - p) dF(v). \end{aligned}$$

This is an optimal-transport problem. By the supermodularity of ϕ_q, the problem is solved by the Fréchet-Hoeffding lower bound

$$\underline{J}(v_L, v_H) := \max\{J^L(v_L) + J^H(v_H) - 1, 0\}$$

(see, e.g., Marshall, Olkin, and Arnold, 2011, Corollary 12.M.3.a). We now state the equivalent p-pairwise information structure: If $v \notin [\underline{v}, \overline{v}]$, display $s = (v, v)$. If $v \in [\underline{v}, \overline{v}]$, display the signal $s = (v_L, v_H) \in [\underline{v}, p] \times [p, \overline{v}]$ that solves

$$v \in \{v_L, v_H\} \quad \text{and} \quad \int_{v_L}^{v_H} (p - v) dF(v) = 0,$$

which is unique because F is strictly increasing.[10] We call this the $(p, \underline{v})$*-negative-assortative* information structure. Using Lemmas 1–4, we have established our main result.

Theorem 1. *For every robust information structure that induces price p, there exists a robust $(p, \underline{v})$-negative-assortative information structure that induces the same price, the same buyer payoff, and a weakly higher seller payoff.*

We conclude this section with an illustration of why negative assortative pooling is most robust. Consider a discrete version of the model in which four valuations $v_1 < v_2 < v_3 < v_4$ have the same probability. Suppose we want to pool them pairwisely such that the posterior valuations is always $p \in (v_2, v_3)$ and negative assortative pooling—v_1 with v_4, v_2 with v_3—would do the trick:

$$p - v_1 = v_4 - p \quad \text{and} \quad p - v_2 = v_3 - p. \tag{7}$$

For robustness, we want to minimize the probability with which the seller can pool the valuations v_1, v_2 into the BUY signal under any q-optimal extension. Observe that if two valuations $v_i < p$ and $v_j > q > p$ are pooled with respective probability ζ_i, ζ_j into a signal such that the posterior valuation equals p, then the q-optimal extension satisfies

$$\zeta_i x_q(v_i, v_j) = \zeta_j \frac{v_j - q}{q - v_i}. \tag{8}$$

The fraction on the right-hand side states how much of the probability of v_i can be pooled into the BUY signal per unit of probability from v_j. This fraction is supermodular.

Suppose the seller considers charging price $q \in (p, v_3)$. Under negative assortative pooling, she can pool v_1, v_2 into the BUY signal with probability

$$\frac{v_4 - q}{q - v_1} + \frac{v_3 - q}{q - v_2}. \tag{9}$$

We compare this with a generic p-pairwise pooling, which is not necessarily negative assortative. Specifically, suppose the valuations are pooled into signals s_1, s_2, s_3, s_4 with the probabilities stated in Table 1, assuming that the posterior valuation is always p:

$$(1 - \zeta_1)(p - v_1) = (1 - \zeta_2)(v_3 - p) \quad \text{and} \quad (1 - \zeta_2)(p - v_2) = (1 - \zeta_1)(v_4 - p). \tag{10}$$

Then, the seller can pool v_1, v_2 into the BUY signal with probability

[10] To see that this is equivalent to $\underline{J}$, note that the support of $\underline{J}$ consists of all pairs (v_L, v_H) such that $J^L(v_L) + J^H(v_H) - 1 = 0$, which is equivalent to $\int_{v_L}^{v_H} (p - v) dF(v) = 0$.

	s_1	s_2	s_3	s_4
v_1	ζ_1	0	$1-\zeta_1$	0
v_2	0	ζ_2	0	$1-\zeta_2$
v_3	0	ζ_2	$1-\zeta_2$	0
v_4	ζ_1	0	0	$1-\zeta_1$

Table 1: Generic p-pairwise pooling

$$\zeta_1 \frac{v_4 - q}{q - v_1} + (1-\zeta_1)\frac{v_4 - q}{q - v_2} + \zeta_2 \frac{v_3 - q}{q - v_2} + (1-\zeta_2)\frac{v_3 - q}{q - v_1}. \tag{11}$$

Subtracting (9) from (11), we get

$$(1-\zeta_1)\left(\frac{v_4 - q}{q - v_2} - \frac{v_4 - q}{q - v_1}\right) - (1-\zeta_2)\left(\frac{v_3 - q}{q - v_2} - \frac{v_3 - q}{q - v_1}\right).$$

By (7) and (10), this difference has the same sign as

$$\phi_q(v_4, v_2) - \phi_q(v_4, v_1) - [\phi_q(v_3, v_2) - \phi_q(v_3, v_1)],$$

which is greater than zero by the supermodularity of ϕ_q. Hence, negative assortative pooling is most robust.

5 Optimal Information Structures

By Theorem 1, the search for buyer-optimal information structures can be restricted to $(p, \underline{v})$-negative-assortative information structures. We now study the remaining problem of choosing the parameters p and $\underline{v}$. Our first goal is to obtain a simple statement of the optimization problem.

A $(p, \underline{v})$-negative-assortative information structure that induces price p yields buyer payoff

$$U(p, \underline{v}) := \int_{\underline{v}}^{1} (v - p) dF(v) \tag{12}$$

and seller payoff $[1 - F(\underline{v})]p$. Because the seller can always extend to perfect information, she must at least get her perfect-information payoff. Let Π^* denote this payoff and let

p^* be the lowest price that the seller is willing to charge under perfect information, that is,

$$\Pi^* := [1 - F(p^*)]p^* \quad \text{and} \quad p^* := \min \operatorname*{argmax}_q [1 - F(q)]q.$$

Consequently, the $(p, \underline{v})$-negative-assortative information structure is robust and induces price p only if $[1 - F(\underline{v})]p \geq \Pi^*$. Moreover, this condition is also sufficient for the $(p, \underline{v})$-negative-assortative information structure to induce p.

The pairs of valuations that are pooled under a $(p, \underline{v})$-negative-assortative information structure are determined by the strictly decreasing function $\mu_p \colon [\underline{v}, \overline{v}] \to [\underline{v}, \overline{v}]$ that is implicitly defined by

$$\mu_p(v) \neq v \quad \text{and} \quad \int_v^{\mu_p(v)} (p - u) dF(u) = 0$$

for $v \neq p$ and $\mu_p(p) = p$. Observe that $\mu_p(\mu_p(v)) = v$ and $\mu_p(\underline{v}) = \overline{v}$. Let the seller's gain in payoff from charging price $q \in [p, \mu_p(\underline{v})]$ and performing the q-optimal extension, rather than charging p and performing no extension, be defined as

$$\Psi(q, p, \underline{v}) := \left[1 - F(q) + \int_{\underline{v}}^{\mu_p(q)} x_q(v, \mu_p(v)) dF(v)\right] q - [1 - F(\underline{v})]p.$$

Robustness requires that the seller's gain Ψ is non-positive for all q.

We conclude that a $(p, \underline{v})$-negative-assortative information structure induces price p and is robust if and only if

$$[1 - F(\underline{v})]p \geq \Pi^*, \tag{13}$$

$$\Psi(q, p, \underline{v}) \leq 0 \quad \text{for all } q \in (p, \mu_p(\underline{v})). \tag{14}$$

Finding a buyer-optimal information structure thus amounts to solving

$$\max_{p, \underline{v}} \; U(p, \underline{v}) \quad \text{s.t. (13) and (14).} \tag{15}$$

We will denote by $(p_B, \underline{v}_B)$ a solution to problem (15).

5.1 Buyer Payoff at the Upper Bound

Since the seller payoff is at least Π^* and the social surplus at most $E[v]$ (obtained when trade happens with probability one), the buyer payoff can never be larger than

$$\overline{U} := \Pi^* - E[v],$$

as stated in Section 3 for the uniform case. We reach this upper bound on the buyer payoff if and only if trade happens with probability one at price $p = \Pi^*$, that is, if the $(\Pi^*, 0)$-negative-assortative information structure is robust. Thus, $(\Pi^*, 0)$ solves (15) if it satisfies (14).

In Section 3, we have shown that the $(\Pi^*, 0)$-negative-assortative information structure *is* robust under the uniform prior. Under many other priors, however, robustness fails, and so the upper bound is not always attainable. We give here a simple necessary condition for robustness of $(\Pi^*, 0)$. Suppose the $(\Pi^*, 0)$-negative-assortative information structure pools valuations $v > p^*$ into posterior valuation Π^*, that is, $\mu_{\Pi^*}(0) > p^*$. Then, robustness fails, for if the seller performs the p^*-optimal extension then the probability of trade at price p^* strictly increases relative to perfect information, resulting in a seller payoff strictly greater than Π^*. Consequently, the $(\Pi^*, 0)$-negative-assortative information structure is robust only if $\mu_{\Pi^*}(0) \leq p^*$. This is equivalent to Condition (16) below since $E[v|v \leq \mu_{\Pi^*}(0)] = \Pi^*$.

Proposition 1. *The* $(\Pi^*, 0)$*-negative-assortative information structure is robust only if the prior* F *satisfies*

$$E[v|v \leq p^*] \geq \Pi^*. \tag{16}$$

The following example identifies a class of priors for which (16) holds and for which numerical simulations suggest that the $(\Pi^*, 0)$-negative-assortative information structure is indeed robust. Priors that do not satisfy (16) are reported in Example 3 in the next subsection.

Example 1. Suppose $F(v) = v^m$ for some $m > 0$. Then, $p^* = (1+m)^{-\frac{1}{m}}$ and

$$E[v|v \leq p^*] = \frac{m}{1+m} p^* = \Pi^*.$$

Hence, (16) holds. Note that for $m = 1$, we have the uniform prior. To determine whether the $(\Pi^*, 0)$-negative-assortative information structures is indeed robust, it remains to check (14). In general, (14) is much less tractable than for the uniform prior. For example, μ_p can typically not be expressed in closed form. It is, however, straightforward to simulate (14) numerically for specific priors. Figure 1 displays $\Psi(q, \Pi^*, 0)$

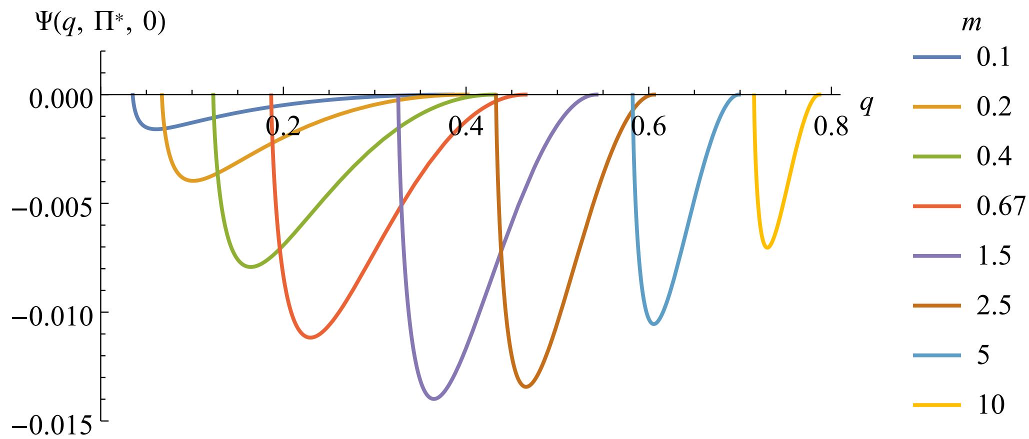

Figure 1: $\Psi(q, \Pi^*, 0)$ plotted for $q \in [\Pi^*, \mu_{\Pi^*}(0)]$ in Example 1, varying parameter m.

as a function of q for eight different values of m, conveying that $\Psi(q, \Pi^*, 0)$ is always non-positive as required by (14).[11] $\triangle$

Condition (16) is not sufficient for the robustness of the $(\Pi^*, 0)$-negative-assortative information structure. In the next example, no price $q \in (\Pi^*, \mu_{\Pi^*}(0))$ yields seller payoff Π^* under perfect information, but some price yields a payoff close to Π^* and the q-optimal extension raises the probability of trade significantly relative to perfect information. As a consequence, robustness fails.

Example 2. Suppose the prior F has the PDF

$$f(v) = \begin{cases} 1 + 4v & \text{if } v \in [0, \frac{1}{8}), \\ \frac{3}{4} & \text{if } v \in [\frac{1}{8}, \frac{3}{8}), \\ 3 - 4v & \text{if } v \in [\frac{3}{8}, \frac{1}{2}), \\ 1 & \text{if } v \in [\frac{1}{2}, 1]. \end{cases}$$

Thus, F coincides with the uniform distribution for $v \geq 1/2$ and is symmetric around $1/4$ for $v < 1/2$. As in the uniform case, $p^* = 1/2$ and $\Pi^* = 1/4 = E[v|v \leq p^*]$. Consider the $(1/4, 0)$-negative-assortative information structure. For $v \leq 1/2$, the symmetry around

[11] All numerical results reported in this paper were obtained using *Mathematica*. The source code is available on request from the authors.

$1/4$ implies $\mu_{1/4}(v) = 1/2 - v$ and

$$x_q(v, \mu_{1/4}(v)) = \frac{\frac{1}{2} - v - q}{q - v},$$

just as in the uniform case. However, the robustness constraint (14) is violated: for $q = 3/8$, one finds

$$\Psi(\tfrac{3}{8}, \tfrac{1}{4}, 0) = \frac{1}{128}\left(13 - 30 \log\left[\frac{3}{2}\right]\right) > 0. \qquad \triangle$$

5.2 When the Upper Bound is Not Attainable

When the $(\Pi^*, 0)$-negative assortative information structure is not robust and the upper bound on the buyer payoff thus unattainable, some progress in solving Problem (15) can be made by narrowing down the choice variables $(p, \underline{v})$. We first show that (14), the robustness constraint, is monotone in $\underline{v}$.

Lemma 5. *(i) $\Psi(q, p, \underline{v})$ is strictly increasing in $\underline{v}$. (ii) If $(p, \underline{v})$ violates* (13) *or* (14)*, then $(p, \underline{v}')$ violates* (13) *or* (14) *for all $\underline{v}' > \underline{v}$.*

If $(\Pi^*, 0)$ is not feasible, then there exists no information structure that implements both trade with probability one and seller payoff Π^*. We will state the best prices that implement *either* of the two and show that optimal prices lie in between. Consider trade with probability one. Then, $\underline{v} = 0$, and the best corresponding price is

$$p_0 := \min\left\{p \geq \Pi^* : \Psi(q, p, 0) \leq 0 \text{ for all } q \in (p, \mu_p(0))\right\}.$$

This price always exists and is weakly smaller than both $E[v]$ and p^*.[12] Next, consider implementing seller payoff Π^*. For any price $p \geq \Pi^*$, denote by

$$\hat{v}(p) := F^{-1}\left(1 - \frac{\Pi^*}{p}\right)$$

the largest $\underline{v}$ such that $(p, \underline{v})$ satisfies (13). Thus, the $(p, \hat{v}(p))$-negative-assortative information structure induces exactly seller payoff Π^*. The best such price is

$$p_1 := \min\left\{p \geq p_0 : \Psi(q, p, \hat{v}(p)) \leq 0 \text{ for all } q \in \Big(p, \mu_p(\hat{v}(p))\Big)\right\}.$$

[12] If $p^* \leq E[v]$, then $(p^*, 0)$ is robust by Lemma 5(i) because (p^*, p^*) represents perfect information and is robust, whereas $(E[v], 0)$ represents no information and is always robust.

This price also always exists, and it is weakly smaller than p^*.[13]

According to the following proposition, p_0 is the lowest implementable price (by Theorem 1, across *all* information structures), and prices above p_1 are dominated for the buyer. This narrows down the choice variables in Problem (15) to $p \in [p_0, p_1]$ and $\underline{v} \in [0, \hat{v}(p_1)]$.

Proposition 2. *(i) There exists no price $p < p_0$ such that $(p, \underline{v})$ satisfies* (13) *and* (14)*.* *(ii) For every $(p, \underline{v})$ that satisfies* (13) *such that $p > p_1$, $U(p, \underline{v}) < U(p_1, \hat{v}(p_1))$.*

We now report numerical results for optimal $(p_B, \underline{v}_B)$ under specific priors. Determining p_0 and p_1 numerically are relatively straightforward one-parameter problems. For any given $p \in (p_0, p_1)$, the best corresponding value for $\underline{v} \in [0, \hat{v}(p)]$ is the lowest value such that the robustness constraint (14) holds (noting Lemma 5(i) and that the buyer payoff strictly increases in $\underline{v}$). Finally, one identifies the optimal p.

Example 2 (continued). We found $p_B \approx 0.27394 \in (p_0, p_1)$ and $\underline{v}_B \approx 0.06211$, resulting in buyer payoff $U(p_B, \underline{v}_B) \approx 0.24294$ and seller payoff $(1 - F(\underline{v}_B))p_B \approx 0.25481$. △

As a final example, we consider a class of priors that does not satisfy (16), the necessary condition for attaining the upper bound.

Example 3. Suppose $F(v) = 1-(1-v)^r$ for some $r \in (0, 1)$. Then, $p^* = E[v] = (1+r)^{-1}$ and $\Pi^* = r^r(1 + r)^{-(1+r)}$, and one can show that (16) does not hold. For several values of r, we have numerically determined p_0, p_1, and the optimal pair $(p_B, \underline{v}_B)$. The results are reported in Table 2, along with the corresponding buyer payoffs and Π^*. In all cases, $\underline{v}_B > 0$ and $p_B \in (p_0, p_1)$. Thus, the optimal information structures for these priors neither result in trade with probability one nor limit the seller payoff to Π^*. △

5.3 A Lower Bound on the Buyer Payoff

We conclude this section with a *lower* bound on the buyer payoff. An optimal information structure must induce at least the perfect-information payoff $\int_{p^*}^1 (v - p^*) dF(v)$, since

[13] Since $(p^*, \hat{v}(p^*))$ represents perfect information and is robust.

r	Π^*	p_0	$U(p_0, 0)$	p_1	$U(p_1, \hat{v}(p_1))$	p_B	$\underline{v}_B$	$U(p_B, \underline{v}_B)$
0.90	0.26865	0.26867	0.25765	0.27090	0.25763	0.26938	0.00299	0.25765
0.75	0.30268	0.30282	0.26860	0.30957	0.26842	0.30485	0.00917	0.26865
0.50	0.38490	0.38586	0.28081	0.40508	0.27933	0.39085	0.02775	0.28109
0.30	0.49546	0.49841	0.27082	0.53511	0.26513	0.50646	0.06036	0.27158
0.25	0.53499	0.53881	0.26119	0.58114	0.25336	0.54758	0.07396	0.26212
0.20	0.58236	0.58724	0.24609	0.63542	0.23537	0.59658	0.09145	0.24721
0.10	0.71527	0.72264	0.18646	0.77909	0.16731	0.73151	0.14438	0.18775
0.05	0.81790	0.82549	0.12689	0.87659	0.10404	0.83178	0.17719	0.12778
0.01	0.94544	0.94915	0.04095	0.97278	0.02590	0.95054	0.16183	0.04109

Table 2: Simulation results for Example 3 with $r < 1$.

perfect information is always robust. We will give conditions under which the buyer can do *strictly* better. In fact, already a relatively simple information structure will improve over perfect information. We need the following lemma.

Lemma 6. *If the prior F satisfies* (16) *or the PDF f is continuous on $(p^* - \epsilon, p^* + \epsilon)$ for some $\epsilon > 0$, then there exists $\tilde{v} < p^*$ such that*

$$[1 - F(\tilde{v})]E[v|v \in [\tilde{v}, p^*]] \geq \Pi^*. \tag{17}$$

Suppose a valuation $\tilde{v}$ as described in the lemma exists. Consider the following information structure: If $v \notin [\tilde{v}, p^*]$, display $s = v$. If $v \in [\tilde{v}, p^*]$, on the other hand, display always the same signal. Hence, if $v \in [\tilde{v}, p^*]$ then the posterior valuation is $E[v|v \in [\tilde{v}, p^*]]$, and otherwise the learning is perfect. By (17), this information structure induces price $E[v|v \in [\tilde{v}, p^*]]$. If it is not robust, then the seller will ultimately charge a price $q \in (\tilde{v}, p^*)$. In either case, the price is strictly smaller than p^* and, therefore, the buyer payoff strictly greater than $\int_{p^*}^1 (v - p^*)dF(v)$. We have thus shown the following.

Proposition 3. *Under the conditions on the prior F in Lemma 6, there exists a robust information structure that induces a buyer payoff strictly greater than $\int_{p^*}^1 (v - p^*)dF(v)$.*

For our main result Theorem 1, we made no restrictions on the information structures and the extensions that the buyer and the seller, respectively, can choose. The negative assortative information structures in the theorem pool at most two valuations with each other and amount to a non-monotone partition of the set of possible valuations. According to the above, these stark features are not necessary to improve over perfect information.

6 Discussion

In this section, we study a weaker robustness constraint, investigate how the seller's ability to add information changes the design problem, and consider a variant of the model where the seller chooses the information structure and the buyer can extend.

6.1 Weak Robustness

So far, we have assumed that the seller's extension can be correlated with the signal that the buyer receives from the original information structure. Such extensions may be the appropriate notion when, for example, the disclosure concerns different product attributes.[14] Nevertheless, the seller's choice of correlated extensions may be limited (after all, the buyer privately observes the original signal). We now discuss how our results change when correlation is impossible.

Formally, if $(S^a, (G^a_v))$ is the original information structure and $(S^a \times S^b, (G_v))$ the extended one, then the extension is *independent* if $G_v = G^a_v G^b_v$ for some CDF G^b_v over S^b. An information structure that provides the seller no incentive for independent extension is *weakly robust*. Every robust information structure is, of course, also weakly robust. In particular, being partitional, a $(p, \underline{v})$-negative-assortative information structure is weakly robust if *and only if* it is robust. All insights in Section 5 on optimal $(p, \underline{v})$-negative-assortative information structures therefore extend without change.

Our main result, Theorem 1, on the other hand, characterizes the implementable buyer payoffs under the robustness constraint. In the derviation, we used the possibility

[14] For illustration, suppose the buyer's valuation can be written $v = \eta(\theta^a, \theta^b)$, the original information structure perfectly disclosing θ^a and the extension θ^b. Abstracting from θ^a and θ^b, we may set $s^a = E[\eta(\theta^a, \theta^b)|\theta^a]$ and $s^b = \eta(\theta^a, \theta^b) - E[\eta(\theta^a, \theta_b)|\theta^a]$. Then unless η is linear, s^a and s^b are correlated.

	$BUY1$	$BUY2$
v_{H1}	1	0
v_{H2}	ρ	$1-\rho$
v_{Li}	$x_q(v_{Li}, v_{H1})$	$(1-\rho)x_q(v_{Li}, v_{H2})$

Table 3: Independent optimal extension

of correlated extensions to show that the restriction to p-pairwise information structures is without loss of generality (Lemma 1) and for the q-optimal extension (Lemma 2). This raises the question of whether certain payoffs are implementable by a weakly robust information structure but not by a robust one. We show here that the respective information structure cannot be p-pairwise.

Proposition 4. *A p-pairwise information structure is robust if and only if it is weakly robust.*

The proof constructs an independent extension that performs as well as the q-optimal extension that conditions on the signal $s = (v_L, v_H)$ and the true valuation v. In particular, there is trade with probability one if $v \geq q$ and with probability $x_q(v_L, v_H)$ if $v = v_L < v_H \in [q, 1]$. For illustration, consider a p-pairwise information structure with just four pooled valuations $v_{L1} < v_{L2} < p < v_{H1} < v_{H2}$. Suppose $q \in (p, v_{H1})$, and set

$$\rho := \frac{x_q(v_{L1}, v_{H1})}{x_q(v_{L1}, v_{H2})} \qquad \left(= \frac{v_{H1} - q}{v_{H2} - q} \frac{v_{H2} - p}{v_{H1} - p} = \frac{x_q(v_{L2}, v_{H1})}{x_q(v_{L2}, v_{H2})} \right).$$

Our independent extension uses two BUY signals, $BUY1$ and $BUY2$. Table 3 gives the likelihoods with which the valuations are pooled into these signals. Both v_{H1} and v_{H2} are pooled with probability one into the BUY signals, and the likelihood ratio of v_{Li} to v_{Hj} in each BUY signal is exactly $x_q(v_{Li}, v_{Hj})$. Hence, the posterior valuation is always q, and we get the same probability of trade as under the q-optimal extension.

Intuitively, this result can be explained as follows. Under a p-pairwise information structure, two distinct valuations v_L, v_H are pooled into at most one signal. To obtain final posterior valuation q (maximizing the probability of trade at price q), the likelihood ratio with which an extension needs to pool v_L, v_H is therefore independent of the signal—the ratio must equal $x_q(v_L, v_H)$.

Under the original robustness constraint, (16) is a necessary condition for the buyer payoff to attain the upper bound $\overline{U}$. We show now that it remains a necessary condition also under weak robustness. We first establish an auxiliary result, which says that valuations above p^* must not be pooled with valuations below. Otherwise, the seller could again trade at price p^* with a greater probability than under perfect information, obtaining a payoff larger than Π^*.

Lemma 7. *An information structure $(S, (G_v))$ that induces buyer payoff $\overline{U}$ is weakly robust only if*

$$\int_{\{s \in S : F_s(p^*) \in (0,1)\}} d\bar{G}(s) > 0. \tag{18}$$

Now, in order to induce trade at price Π^* with probability one and thus to attain the upper bound $\overline{U}$, the lowest posterior valuation must be at least Π^*. By Lemma 7, the valuations below p^* must consequently be pooled in such a way that the lowest posterior valuation is at least Π^*. This is possible only if if the prior mean of these valuations is greater than Π^*, which is Condition (16).

Proposition 5. *Weakly robust information structures that induce buyer payoff $\overline{U}$ exist only if the prior F satisfies* (16)*.*

6.2 Comparison: No Disclosure by the Seller

Here, we compare our results with those of Roesler and Szentes (2017), who study buyer-optimal information structures when the seller cannot disclose more. They identify a class of information structures, from now on called *RS class*, with the property that for every information structure there exists one in this class that generates the same seller payoff and a weakly higher buyer payoff. Without the possibility of disclosure by the seller, the only relevant property of an information structure is the induced CDF of posterior valuations. The RS class is characterized by the family of CDFs

$$H_q^B(w) = \begin{cases} 0 & \text{if } w \in [0, q), \\ 1 - \frac{q}{w} & \text{if } w \in [q, B), \\ 1 & \text{if } w \in [B, 1] \end{cases}$$

for $q \in (0,1]$ and $B \in [q,1]$. Among these CDFs, only those for which the prior F is a mean-preserving spread can indeed by induced by some information structure. Observe that for a given H_q^B, the seller is indifferent which price $p \in [q,B]$ to charge. Hence, her payoff is q, and trade happens with probability one if she charges price q.

According to Roesler and Szentes (2017, Theorem 1), the buyer-optimal information structures in their setting result in trade with probability one at price

$$p^{RS} := \min\left\{q : \exists B \in [q,1] \text{ s.t. } F \text{ is a mean-preserving spread of } H_q^B\right\}.$$

It turns out that none of these information structures is (weakly) robust.

Proposition 6. $p^{RS} < \Pi^*$. *Hence, the information structures that are buyer-optimal when the seller cannot provide an extension are not weakly robust.*

Consequently, the (weak) robustness constraint binds for all priors F, and the seller's ability to add information always makes the buyer strictly worse off and the seller strictly better off.

Roesler and Szentes (2017, Lemma 1) show that if some information structure results in seller payoff q, then the CDF of posterior valuations is a mean-preserving spread of the corresponding CDF H_q^B. In this sense, the information structures in the RS class are least informative. In the present setting, in contrast, the goal is to implement the desired CDF of posterior valuations with an information structure that is *as* informative (hence, as robust) as possible. This suggests that the restriction to the RS class may not be without loss of generality when the seller can add information. Indeed, according to the following result, the upper bound $\overline{U}$ on the buyer payoff in our setting is never attainable with the RS class.

Proposition 7. *There is no weakly robust information structure in the RS class that induces buyer payoff* $\overline{U}$.

6.3 Reversed Timing

Suppose the *seller* chooses the information structure and the *buyer* can extend. When deciding whether to extend, the buyer knows the information structure but not yet the signal. The seller, in turn, knows the extension, if any, when she sets the price. One can interpret this model variant such that the extension is actually performed by a consumer protection agency, which reacts to the seller's disclosure.

When we examine the incentives to extend, we assume that the seller always sets the lowest price that is optimal for her. We adapt our terminology and say that an information structure induces price p (and the corresponding buyer and seller payoff) if

$$p = \min \underset{q}{\operatorname{argmax}} [1 - H(q) - \Delta(H, q)]q.$$

An information structure is *buyer robust* if the buyer has no incentive to extend it. Analogously to the original model, we can restrict attention to buyer-robust information structures. Notice that the seller payoff must again be at least Π^*, the maximum payoff under a perfect information structure, as the seller can always provide perfect information. We will show that the seller payoff must actually be equal to Π^*.

Virtually the same argument as in the original model allows to confine the analysis to p-pairwise information structures (cf. Lemma 1).

Lemma 8. *Every buyer-robust information structure that induces price p and seller payoff $\Pi \geq \Pi^*$ can be extended to a p-pairwise information structure that induces the same price, the same buyer payoff, and the same seller payoff.*

Consider any p-pairwise information structure that induces price p and a seller payoff *strictly* greater than Π^*. Suppose the buyer additionally learns whether or not his valuation is below some cutoff $v' < p$. Thus, he learns the valuation perfectly for signals $s = (v_L, v_H)$ with $v_L < v' < v_H$, whereas for all other signals he learns nothing new. It is not hard to see that there exists a cutoff v' such that, at price p, this extension strictly decreases the probability of trade but the seller still obtains at least payoff Π^*. As the extended information structure is still p-pairwise, the seller payoff at prices $q > p$ is still at most $[1 - F(q)]q \leq \Pi^*$. Hence, the seller will charge a price $q \leq p$. But then, the original information structure was not buyer robust. This establishes the following proposition.

Proposition 8. *Every buyer-robust information structure that is optimal for the seller induces seller payoff Π^*.*

Perhaps surprisingly, the seller does not have a first-mover advantage: under the original timing, where the buyer chooses the information structure and the seller can extend, the seller payoff may be larger than Π^* (see Example 3). Intuitively, the possibility to extend gives the buyer a more direct influence on the seller's choice of the price than the design of the information structure. The buyer may indeed prefer the reversed timing. For example, suppose (16) holds. Then, the seller is willing to choose the $(\Pi^*, 0)$-negative assortative information structure, for she can secure payoff Π^* by charging price p^*. Hence, unlike under the original timing, (16) is sufficient for attaining the upper bound on the buyer payoff $\overline{U}$.

According to Proposition 8, there always exists an equilibrium with full disclosure. Generally, this will not be the only equilibrium, though. Whereas the seller must always get the same payoff Π^*, the buyer is typically not indifferent which equilibrium is played.

7 Conclusion

The goal of this paper was to study buyer-optimal information structures when the seller can disclose more. To prevent such disclosure, the design problem includes the constraint that the information structure must be robust. The most robust information structures pool the buyer's valuation in a deterministic, negative assortative fashion: the negative assortative information structures of Theorem 1 implement every implementable buyer payoff. Indeed, they also implement the highest possible corresponding seller payoff and every implementable price. Hence, if the designer seeks to maximize any increasing function of buyer and seller payoff, possibly subject to price constraints, attention can be restricted to negative assortative information structures. We also uncovered a connection between information design and matching, or optimal transport. Suppose a given set of states is to be pooled pairwisely into a given posterior mean. As we have shown, the class of all such poolings is equivalent to a certain class of all bivariate distributions with given marginals, and hence any problem of finding an optimal pooling is an optimal-transport problem. Whenever not just the distribution of posterior means matters, but also *how* states are pooled into posterior means, this connection might be useful.

A Appendix: Proofs

Proof of Lemma 1. Let $(S^a, (G_v^a))$ be robust. We first extend $(S^a, (G_v^a))$ such that the support of the posterior belief consists of at most two valuations almost surely and the CDF of posterior valuations remains unchanged. The extended information structure, denoted by $(S^{ab}, (G_v^{ab}))$, has signals (s^a, s^b), where $s^b \in S^b = [0,1]^2$. In the following, we define the CDF over s^b conditional on v and s^a, assuming without loss of generality that the support of F_{s^a} is not a singleton. Let

$$\begin{aligned} w(s^a) &:= E[v|s^a], \\ c(s^a) &:= \int_0^{w(s^a)} (w(s^a) - v)dF_{s^a}(v) = \int_{w(s^a)}^1 (v - w(s^a))dF_{s^a}(v). \end{aligned}$$

We write $s^b = (v_L, v_H)$, where $v_L \le v_H$. If $v \in [0, w(s^a)]$, then (v_L, v_H) is drawn from the set $\{(v_L, v_H) : v_L = v, v_H \in [w(s^a), 1]\}$ according to the CDF

$$G_v(v_H|s^a) = \frac{1}{c(s^a)} \int_{w(s^a)}^{v_H} (u_H - w(s^a))dF_{s^a}(u_H). \tag{A.1}$$

If $v \in [w(s^a), 1]$, on the other hand, (v_L, v_H) is drawn from the set $\{(v_L, v_H) : v_L \in [0, w(s^a)], v_H = v\}$ according to the CDF

$$G_v(v_L|s^a) = \frac{1}{c(s^a)} \int_0^{v_L} (w(s^a) - u_L)dF_{s^a}(u_L).$$

The distribution function of (v_L, v_H) conditional on s^a is thus given by

$$\begin{aligned} \bar{G}(v_L, v_H|s^a) = &\quad \frac{1}{c(s^a)} \int_0^{v_L} \int_{w(s^a)}^{v_H} (u_H - w(s^a))dF_{s^a}(u_H)dF_{s^a}(u_L) \\ &+ \frac{1}{c(s^a)} \int_{w(s^a)}^{v_H} \int_0^{v_L} (w(s^a) - u_L)dF_{s^a}(u_L)dF_{s^a}(u_H) \\ = &\quad \frac{1}{c(s^a)} \int_0^{v_L} \int_{w(s^a)}^{v_H} (u_H - u_L)dF_{s^a}(u_H)dF_{s^a}(u_L), \end{aligned} \tag{A.2}$$

where the last line follows from Fubini's Theorem. Clearly, under the extended information structure the support of the posterior belief consists of at most two valuations almost surely. Specifically, the posterior belief $F_{s^a,(v_L,v_H)}$ has support $\{v_L, v_H\}$ and is characterized by the probability $F_{s^a,(v_L,v_H)}(v_L)$ that the valuation equals v_L. For $M \in \mathcal{B}([0, w(s^a)] \times [w(s^a), 1])$, let M_L be the projection on $[0, w(s^a)]$ and M_H the projection on $[w(s^a), 1]$. Analogously to the definition of the posterior belief in (1),

$$\int_M F_{s^a,(v_L,v_H)}(v_L)d\bar{G}(v_L, v_H|s^a) = \int_{M_L} \int_{M_H} dG_{v_L}(u_H|s^a)dF_{s^a}(v_L). \tag{A.3}$$

Plugging (A.1) and (A.2) into (A.3) gives

$$\begin{aligned} & \frac{1}{c(s^a)} \int_{M_L} \int_{M_H} F_{s^a,(v_L,v_H)}(v_L)(v_H - v_L) dF_{s^a}(v_H) dF_{s^a}(v_L) \\ = \quad & \frac{1}{c(s^a)} \int_{M_L} \int_{M_H} (v_H - w(s^a)) dF_{s^a}(v_H) dF_{s^a}(v_L). \end{aligned}$$

Since this equation holds for $F_{s^a,(v_L,v_H)}(v_L) = (v_H - w(s^a))/(v_H - v_L)$, and since $F_{s^a,(v_L,v_H)}$ is unique for almost all (v_L, v_H), we have $E[v|s^a,(v_L,v_H)] = w(s^a)$ almost surely. Thus, the extended information structure $(S^{ab},(G_v^{ab}))$ induces the same CDF of posterior valuations as $(S^a,(G_v^a))$. Consequently, $(S^{ab},(G_v^{ab}))$ induces the same price, the same buyer payoff, and the same seller payoff. Moreover, $(S^{ab},(G_v^{ab}))$ is also robust, because the seller could have performed the extension herself.

Let p be any optimal price for the seller under $(S^{ab},(G_v^{ab}))$. We now extend $(S^{ab},(G_v^{ab}))$ to a p-pairwise information structure $(S^{abc},(G_v^{abc}))$. Conditional on signal $(s^a,(v_L,v_H))$, the extension acts as follows:

- If $E[v|s^a,(v_L,v_H)] = p$, then $E[v|s^a,(v_L,v_H),s^c] = p$ (no disclosure).
- If $E[v|s^a,(v_L,v_H)] > p$ and $v_L < p < v_H$, then $E[v|s^a,(v_L,v_H),s^c] \in \{p, v_H\}$ (partial disclosure).
- In all other cases, $E[v|s^a,(v_L,v_H),s^c] \in \{v_L, v_H\}$ (full disclosure).

Clearly, the such extended information structure is p-pairwise. Note that by the robustness of $(S^{ab},(G_v^{ab}))$, signals $(s^a,(v_L,v_H))$ with $E[v|s^a,(v_L,v_H)] < p$ and $v_L < p < v_H$ have zero probability. Hence, by construction, $E[v|s^a,(v_L,v_H),s^c] \geq p$ if and only if $E[v|s^a,(v_L,v_H)] \geq p$, and so at price p, the buyer payoff and the probability of trade remain unchanged. By the latter, also the seller payoff remains unchanged at p. Since $(S^{ab},(G_v^{ab}))$ was robust, it follows that p remains optimal for the seller and that $(S^{abc},(G_v^{abc}))$ is robust as well. $\square$

Proof of Lemma 2. In the main text. $\square$

Proof of Lemma 3. Let $(S,(G_v))$ be p-pairwise. By (3) and (4), for $s = (v_L, v_H)$ with $v_L < v_H$ we have almost surely

$$\int_0^1 v dF_s(v) = p. \tag{A.4}$$

Hence, for every measurable set M of such signals,

$$\int_M \int_0^1 v dF_s(v) d\bar{G}(s) = p \int_M d\bar{G}(s),$$

which implies

$$\int_M \int_p^1 (v-p) dF_s(v) d\bar{G}(s) = \int_M \int_0^p (p-v) dF_s(v) d\bar{G}(s).$$

Using the definition of the posterior F_s in (1), we obtain

$$\int_p^1 \int_M (v-p) dG_v(s) dF(v) = \int_0^p \int_M (p-v) dG_v(s) dF(v). \tag{A.5}$$

Now let $M = [0, v_L] \times [p, v_H]$ for $0 \leq v_L \leq p \leq v_H \leq 1$. (A.5) can then be written as

$$\int_p^{v_H} \int_0^{v_L} dG_v^L(u)(v-p) dF(v) = \int_0^{v_L} \int_p^{v_H} dG_v^H(u)(p-v) dF(v). \tag{A.6}$$

Hence, we can set

$$c = \int_p^1 G_v^L(p)(v-p) dF(v) = \int_0^p [1 - G_v^H(p)](p-v) dF(v).$$

Then,

$$J(v_L, v_H) = \frac{1}{c} \int_0^{v_L} \int_p^{v_H} dG_v^H(u)(p-v) dF(v)$$

is a bivariate distribution function on $[0, p] \times [p, 1]$ with marginals

$$\begin{aligned} J(v_L, 1) &= \frac{1}{c} \int_0^{v_L} [1 - G_v^H(p)](p-v) dF(v), \\ J(p, v_H) &= \frac{1}{c} \int_p^{v_H} G_v^L(p)(v-p) dF(v). \end{aligned}$$

Now let J be p-pairwise with given α. We construct an equivalent p-pairwise information structure $(S, (G_v))$. For $v < p$, the support of $G_v(v, v_H) = G_v^H(v_H)$ is contained in $\{v\} \times (\{v\} \cup [p, 1])$. Specifically, $G_v^H(p) = 1 - \alpha(v)$ and

$$\begin{aligned} J(v_L, v_H) &= \frac{1}{c} \int_0^{v_L} \int_p^{v_H} d\left(\frac{G_v^H(u) - G_v^H(p)}{\alpha(v)}\right) \alpha(v)(p-v) dF(v) \\ &= \frac{1}{c} \int_0^{v_L} \int_p^{v_H} dG_v^H(u)(p-v) dF(v). \end{aligned}$$

Hence, the CDFs $[G_v^H(u) - G_v^H(p)]/\alpha(v)$ of u on $[p, 1]$ are the CDFs corresponding to a regular conditional distribution, which exists and is unique almost everywhere (see, e.g., Dudley, 2002, Thm. 10.2.2). Analogously, for $v > p$ the support of $G_v(v_L, v) = G_v^L(v_L)$ is contained in $([0,p] \cup \{v\}) \times \{v\}$, and we have $G_v^L(p) = \alpha(v)$ and

$$\begin{aligned} J(v_L, v_H) &= \frac{1}{c} \int_p^{v_H} \int_0^{v_L} d\left(\frac{G_v^L(u)}{\alpha(v)}\right) \alpha(v)(v-p)dF(v) \\ &= \frac{1}{c} \int_p^{v_H} \int_0^{v_L} dG_v^L(u)(v-p)dF(v). \end{aligned}$$

This gives (A.6). Proceeding as in the first part of the proof but in reverse order to (A.4), (A.6) can be transformed into

$$\int_0^{v_L} \int_p^{v_H} E[v-p|s]d\bar{G}(s) = 0. \tag{A.7}$$

To show that $(S, (G_v))$ is p-pairwise, we use (A.7) to show that (4) holds for almost all signals $s = (v_L, v_H) \in [0,p] \times [p,1]$. A function whose integral is zero on every measurable set is zero almost everywhere (see, e.g., Rudin, 1987, Thm. 1.39(b)). Since the probability measure corresponding to $\bar{G}$ is regular (cf, e.g., Rudin, 1987, Thm. 2.18), every measurable set in $[0,p] \times [p,1]$ can be approximated by a countable union of closed balls. By (A.7) and since $\bar{G}$ is atomless, the integral of $E[v-p|s]$ with respect to $\bar{G}$ is zero on every closed ball. Hence, $E[v-p|s] = 0$ for almost all $s \in [0,p] \times [p,1]$. $\square$

Proof of Lemma 4. Let J be a robust p-pairwise distribution function that induces price p. Denote by C the copula of J, that is, $J(v_L, v_H) = C(J^L(v_L), J^H(v_H))$. For $\underline{v}$ such that

$$\int_{\underline{v}}^p (p-v)dF(v) = \int_0^p \alpha(v)(p-v)dF(v) = c, \tag{A.8}$$

let $\widetilde{J}$ be a $(p, \underline{v})$-pairwise distribution function that also has copula C, which exists by Sklar's Theorem. If $\widetilde{J}$ induces price p, the buyer payoff (5) is $\int_p^1 (v-p)dF(v) - c$ as under J, and the seller payoff (6) is weakly higher under $\widetilde{J}$ than under J because (A.8) implies

$$\int_{\underline{v}}^p dF(v) \geq \int_0^p \alpha(v)dF(v).$$

Observe that for $v_L \in [0, \underline{v}]$, we have $J^L(v_L) \geq 0 = \widetilde{J}^L(v_L)$, whereas for $v_L \in [\underline{v}, p]$,

$$1 - J^L(v_L) = \frac{1}{c} \int_{v_L}^p \alpha(v)(p-v)dF(v) \leq \frac{1}{c} \int_{v_L}^p (p-v)dF(v) = 1 - \widetilde{J}^L(v_L).$$

Moreover, for $v_H \in [p, \overline{v}]$,

$$J^H(v_H) = \frac{1}{c}\int_p^{v_H} \alpha(v)(v-p)dF(v) \le \frac{1}{c}\int_p^{v_H} (v-p)dF(v) = \widetilde{J}^H(v_H),$$

whereas for $v_H \in [\overline{v}, 1]$, $J^H(v_H) \le 1 = \widetilde{J}^H(v_H)$. Consequently,

$$J^L(v_L) \ge \widetilde{J}^L(v_L) \text{ for all } v_L \qquad \text{and} \qquad J^H(v_H) \le \widetilde{J}^H(v_H) \text{ for all } v_H. \tag{A.9}$$

For any p-pairwise distribution function $\widehat{J}$, the difference in the seller's payoff when she charges price $q \neq p$ and performs the q-optimal extension, rather than charging p and disclosing no further information, equals

$$\begin{aligned}&\left[\int_S \phi_q(v_L, v_H)d\widehat{J}(v_L, v_H) + 1 - F(q)\right] q - \left[\int_S \phi_p(v_L, v_H)d\widehat{J}(v_L, v_H) + 1 - F(p)\right] p \\ &= \int_S [\phi_q(v_L, v_H)q - \phi_p(v_L, v_H)p]d\widehat{J}(v_L, v_H) + [1 - F(q)]q - [1 - F(p)]p.\end{aligned}$$

We will show that the integral is smaller for $\widehat{J} = \widetilde{J}$ than for $\widehat{J} = J$. Accordingly, since J is robust and induces price p, so does $\widetilde{J}$. We will use the shorthand notation

$$\delta(v_L, v_H) := \phi_q(v_L, v_H)q - \phi_p(v_L, v_H)p = \max\left\{\frac{q(v_H - q)c}{(v_H - p)(q - v_L)}, 0\right\} - \frac{pc}{(p - v_L)}.$$

Straightforward calculus shows that δ is decreasing in v_L and increasing v_H.

Define $\overline{v}_L := p - v_L$. Let K be the joint distribution function of $\overline{v}_L$ and v_H that is implied by J. The marginals of K are $K^L(\overline{v}_L) = 1 - J^L(p - \overline{v}_L)$ and $K^H(v_H) = J^H(v_H)$. Let D be the copula of K and recall that C is the copula of J. By Nelsen (2006, Thm. 2.4.4), $D(u_1, u_2) = u_2 - C(1 - u_1, u_2)$. Let $\widetilde{K}$, $\widetilde{K}^L$, and $\widetilde{K}^H$ be the corresponding distribution functions implied by $\widetilde{J}$. Note that $\widetilde{K}$ also has copula D.

Because of (A.9), we have $K^L(\overline{v}_L) \le \widetilde{K}^L(\overline{v}_L)$ for all $\overline{v}_L$ and $K^H(v_H) \le \widetilde{K}^H(v_H)$ for all v_H. Together with the fact that K and $\widetilde{K}$ have a common copula, this implies according to Shaked and Shanthikumar (2007, Thm. 6.B.14) that $\widetilde{K}$ is smaller than K in the usual stochastic order. Hence, since $\delta(p - \overline{v}_L, v_H)$ is increasing in $\overline{v}_L$ and v_H,

$$\begin{aligned}\int_0^p \int_p^1 \delta(v_L, v_H)d\widetilde{J}(v_L, v_H) &= \int_0^p \int_p^1 \delta(p - \overline{v}_L, v_H)d\widetilde{K}(\overline{v}_L, v_H) \\ &\le \int_0^p \int_p^1 \delta(p - \overline{v}_L, v_H)dK(\overline{v}_L, v_H) \\ &= \int_0^p \int_p^1 \delta(v_L, v_H)dJ(v_L, v_H). \qquad \square\end{aligned}$$

Proof of Theorem 1. In the main text. $\square$

Proof of Proposition 1. In the main text. $\square$

Proof of Lemma 5. (i) Ψ is strictly increasing since for any $v_1 < v_2$ and $q \in (p, \mu_p(v_2))$,

$$\Psi(q,p,v_2) - \Psi(q,p,v_1) = \int_{v_1}^{v_2} p - qx_q(v, \mu_p(v))dF(v) > 0,$$

where the inequality follows from

$$x_q(v, \mu_p(v)) = \frac{(p-v)[\mu_p(v) - q]}{(q-v)[\mu_p(v) - p]} < \frac{p-v}{q-v} \leq \frac{p}{q}.$$

(ii) Consider $\underline{v}' > \underline{v}$. Obviously, if $(p, \underline{v})$ violates (13), then so does $(p, \underline{v}')$. Hence, suppose $(p, \underline{v})$ violates (14), that is, there exists $q \in (p, \mu_p(\underline{v}))$ such that $\Psi(q, p, \underline{v}) > 0$. Note that $\mu_p(\underline{v}') < \mu_p(\underline{v})$. Now, if $q \in (p, \mu_p(\underline{v}'))$, then $\Psi(q, p, \underline{v}') > \Psi(q, p, \underline{v})$ implies that $(p, \underline{v}')$ violates (14). If $q \in [\mu_p(\underline{v}'), \mu_p(\underline{v}))$, which is equivalent to $\mu_p(q) \in (\underline{v}, \underline{v}']$, then

$$0 < \Psi(q,p,\underline{v}) < \Psi(q,p,\mu_p(q)) = [1 - F(q)]q - [1 - F(\mu_p(q))]p \leq \Pi^* - [1 - F(\underline{v}')]p$$

and thus $(p, \underline{v}')$ violates (13). $\square$

Proof of Proposition 2. (i) For $p < \Pi^*$, (13) cannot hold. Hence, suppose $\Pi^* < p_0$ and consider a price $p \in [\Pi^*, p_0)$. By the definition of p_0, for any such p there exists $q \in (p, \mu_p(0))$ such that $\Psi(q, p, 0) > 0$, that is, $(p, 0)$ violates (14). By Lemma 5(ii), $(p, \underline{v})$ violates (13) or (14) for any $\underline{v}$.

(ii) Suppose $(p, \underline{v})$ with $p > p_1$ satisfies (13). Then $\underline{v} \leq \hat{v}(p)$, and so

$$U(p, \underline{v}) \leq U(p, \hat{v}(p)) = \int_{\hat{v}(p)}^{1} vdF(v) - \Pi^* < \int_{\hat{v}(p_1)}^{1} vdF(v) - \Pi^* = U(p_1, \hat{v}(p_1)),$$

where the first inequality holds because $U(p, \underline{v})$ is strictly increasing in $\underline{v}$ and the second inequality because $\hat{v}(p)$ is strictly increasing in p. $\square$

Proof of Lemma 6. Under (16) this is clear. So let $\epsilon > 0$ be such that f is continuous on $(p^* - \epsilon, p^* + \epsilon)$. For $\tilde{v}$ in that interval, define $\Omega(\tilde{v}) := [1 - F(\tilde{v})]E[v|v \in [\tilde{v}, p^*]]$. We show that the derivative Ω' exists, is continuous, and satisfies $\Omega'(p^*) < 0$. Noting that $\Omega(p^*) = \Pi^*$, this will imply the existence of $\tilde{v} < p^*$ with $\Omega(\tilde{v}) \geq \Pi^*$.

Using integration by parts,

$$E[v|v \in [\tilde{v}, p^*]] = \frac{\int_{\tilde{v}}^{p^*} v f(v) dv}{F(p^*) - F(\tilde{v})} = \tilde{v} + \frac{\int_{\tilde{v}}^{p^*} (F(p^*) - F(v)) dv}{F(p^*) - F(\tilde{v})}.$$

By the continuity of f, the derivative exists everywhere on $(p^* - \epsilon, p^* + \epsilon)$ and equals

$$\frac{d}{d\tilde{v}} E[v|v \in [\tilde{v}, p^*]] = f(\tilde{v}) \frac{\int_{\tilde{v}}^{p^*} (F(p^*) - F(v)) dv}{[F(p^*) - F(\tilde{v})]^2} < f(\tilde{v}) \frac{p^* - \tilde{v}}{F(p^*) - F(\tilde{v})},$$

which gives

$$\Omega'(\tilde{v}) < -f(\tilde{v}) E[v|v \in [\tilde{v}, p^*]] + [1 - F(\tilde{v})] f(\tilde{v}) \frac{p^* - \tilde{v}}{F(p^*) - F(\tilde{v})}.$$

By the continuity of f, Ω' is continuous as well on $(p^* - \epsilon, p^* + \epsilon)$, and so

$$\Omega'(p^*) = \lim_{\tilde{v} \to p^*} \Omega'(\tilde{v}) < -f(p^*) p^* + 1 - F(p^*) = 0.$$

For the last equality, we have used that $p^* = \min \operatorname{argmax}_p [1 - F(p)] p$ satisfies the first-order condition $f(p^*) p^* = 1 - F(p^*)$. □

Proof of Proposition 3. In the main text. □

Proof of Proposition 4. We will show that for any p-pairwise information structure it is possible to construct an independent extension that induces same probability of trade at price q, and hence the same seller payoff, as the q-optimal extension. Consequently, robustness and weak robustness are equivalent.

Let $(S^a, (G_v^a))$ be p-pairwise. For prices $q < p$, the q-optimal extension is to disclose nothing, so choose $q \in (p, 1]$. Regardless of the extension, there will be trade with probability one after all signals $s^a = (v_L, v_H)$ for which $v_L = v_H \geq q$. Moreover, there will be no trade after all s^a with $v_H < q$. Only after signals s^a with $v_L < q \leq v_H$, the probability of trade depends on the extension. We denote the set of such signals by $\widehat{S}^a := \{s^a \in S^a : v_L < q \leq v_H\}$.

Consider the following independent extension. The second signal component is drawn from $S^b = [q, 1]$. The CDFs G_v^b condition on v such that:

- If $v \in [0, p)$, then

$$G_v^b(s^b) = \frac{p - v}{q - v} \frac{s^b - q}{s^b - p} + 1 - \frac{p - v}{q - v} \frac{1 - q}{1 - p}.$$

Hence, G_v^b has an atom at $s^b = q$, support $[q,1]$, and for $s^b \in (q,1]$ a PDF g_v^b with

$$g_v^b(s^b) = \frac{p-v}{q-v}\frac{q-p}{(s^b-p)^2}.$$

- If $v \in [q,1]$, then

$$G_v^b(s^b) = \frac{v-p}{v-q}\frac{s^b-q}{s^b-p}.$$

Hence, G_v^b is atomless, has support $[q,v]$, and a PDF g_v^b with

$$g_v^b(s^b) = \frac{v-p}{v-q}\frac{q-p}{(s^b-p)^2}.$$

Conditional on $s^a = (v_L, v_H) \in \widehat{S}^a$ and $s^b \in (q, v_H]$, there is trade with probability one since

$$\begin{aligned} E[v|s^a, s^b] &= \frac{F_{s^a}(v_L)\, g_{v_L}^b(s^b)\, v_L + [1-F_{s^a}(v_L)]\, g_{v_H}^b(s^b)\, v_H}{F_{s^a}(v_L)\, g_{v_L}^b(s^b) + [1-F_{s^a}(v_L)]\, g_{v_H}^b(s^b)} \\ &= \frac{F_{s^a}(v_L)\frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L}v_L + [1-F_{s^a}(v_L)]v_H}{F_{s^a}(v_L)\frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L} + 1 - F_{s^a}(v_L)} \\ &= \frac{F_{s^a}(v_L)x_q(v_L,v_H)v_L + [1-F_{s^a}(v_L)]v_H}{F_{s^a}(v_L)x_q(v_L,v_H) + 1 - F_{s^a}(v_L)} = q. \end{aligned}$$

Consequently, there is also trade with probability one conditional on $s^a \in \widehat{S}^a$ and $v = v_H$. Conditional on $s^a \in \widehat{S}^a$ and $v = v_L$, on the other hand, there is trade if $s^b \in (q, v_H]$, which happens with probability

$$G_{v_L}^b(v_H) - G_{v_L}^b(q) = \frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L} = x_q(v_L, v_H).$$

By Lemma 2, we thus have the same probability of trade as under the q-optimal extension. □

Proof of Lemma 7. Suppose (18) holds. By the right-continuity of distribution functions, there exists a $\delta > 0$ and a subset of signals

$$M := \{s \in S : 0 < F_s(p^*) \text{ and } F_s(p^*+\delta) < 1\}$$

such that $\int_M d\bar{G}(s) > 0$. We will construct an independent extension of $(S, (G_v))$ that induces trade at price p^* with a probability strictly larger than $1-F(p^*)$, and thus yields the seller a payoff strictly larger than Π^*.

Consider the following independent extension. If $v \in (p^*, p^* + \delta]$, display a signal $BUY1$ with probability one. If $v > p^* + \delta$, display $BUY2$ with probability one. If $v \leq p^*$, display $BUY2$ with some probability $\epsilon > 0$ and otherwise $\neg BUY$. Hence, the posterior valuation given s and $BUY2$ is

$$\begin{aligned} w_\epsilon(s, BUY2) := \quad & \frac{\epsilon F_s(p^*)}{\epsilon F_s(p^*) + 1 - F_s(p^* + \delta)} E[v|s, v \leq p^*] \\ & + \frac{1 - F_s(p^* + \delta)}{\epsilon F_s(p^*) + 1 - F_s(p^* + \delta)} E[v|s, v > p^* + \delta]. \end{aligned}$$

Let $M_\epsilon := \{s \in M : w_\epsilon(s, BUY2) < p^*\}$. The probability of trade given price p^* is then at least

$$Q := 1 - F(p^*) + \left[\int_M F_s(p^*) d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*) d\bar{G}(s)\right] \epsilon - \int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s).$$

By the definition of M_ϵ,

$$\begin{aligned} & E[w_\epsilon(s, BUY2)|s \in M_\epsilon] \\ = \quad & \frac{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s)}{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) + \int_{M_\epsilon} 1 - F_s(p^* + \delta) d\bar{G}(s)} E[v|s \in M_\epsilon, v \leq p^*] \\ & + \frac{\int_{M_\epsilon} 1 - F_s(p^* + \delta) d\bar{G}(s)}{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) + \int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s)} E[v|s \in M_\epsilon, v > p^* + \delta] \\ < \quad & p^*, \end{aligned}$$

which implies

$$\begin{aligned} \int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s) &< \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \frac{p^* - E[v|s \in M_\epsilon, v \leq p^*]}{E[v|s \in M_\epsilon, v > p^* + \delta] - p^*} \epsilon \\ &\leq \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \frac{p^*}{\delta} \epsilon. \end{aligned}$$

Consequently,

$$Q > 1 - F(p^*) + \left[\int_M F_s(p^*) d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \left(1 + \frac{p^*}{\delta}\right)\right] \epsilon.$$

Now, $\lim_{\epsilon \to 0} w_\epsilon(s, BUY2) = E[v|s, v > p^* + \delta] > p^*$. By Ergorov's Theorem, it follows that

$$\lim_{\epsilon \to 0} \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) = 0.$$

On the other hand, $\int_M F_s(p^*)d\bar{G}(s) > 0$. Hence, there exist $\epsilon > 0$ such that

$$\int_M F_s(p^*)d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*)d\bar{G}(s)\left(1 + \frac{p^*}{\delta}\right) > 0$$

and thus $Q > 1 - F(p^*)$. □

Proof of Proposition 5. Consider any weakly robust information structure $(S, (G_v))$ that induces buyer payoff $\overline{U}$, that is, trade with probability one at price Π^*. The CDF H of posterior valuations thus satisfies $H(w) = 0$ for all $w < \Pi^*$. Then,

$$\begin{aligned}
& \int_0^{p^*} w dH(w) \geq H(p^*)\Pi^* \\
\iff \quad & \int_{\{s \in S: F_s(p^*)=1\}} \int_0^1 v dF_s(v) d\bar{G}(s) \geq \int_{\{s \in S: F_s(p^*)=1\}} \int_0^1 dF_s(v) d\bar{G}(s)\Pi^* \\
\iff \quad & \int_S \int_0^{p^*} v dF_s(v) d\bar{G}(s) \geq \int_S \int_0^{p^*} dF_s(v) d\bar{G}(s)\Pi^* \\
\iff \quad & \int_0^{p^*} v dF(v) \geq F(p^*)\Pi^*,
\end{aligned}$$

where the second line follows from Lemma 7 and the last line from the definition of the posterior in (1). □

Proof of Proposition 6. According to Roesler and Szentes (2017, Corollary 1), $p^{RS} \leq \Pi^*$. In the following we will show that this inequality is strict. Consequently, any information structure that is buyer optimal in the setting of Roesler and Szentes is not weakly robust: the seller can increase her payoff from p^{RS} to Π^* by independently extending to a perfect information structure.

We start with an auxiliary result. According to Roesler and Szentes (2017, Lemma 1), there is a unique $B^* \in [\Pi^*, 1]$ such that F is a mean-preserving spread of $H_{\Pi^*}^{B^*}$. We strengthen this as follows.

Claim 1. $B^* \in (\Pi^*, 1)$.

Proof. Note that $\Pi^* = \max_p [1 - F(p)]p$ implies $0 < \Pi^* < \int_0^1 v dF(v)$.

As $[1 - F(w)]w \leq \Pi^*$, we have $1 - \frac{\Pi^*}{w} \leq F(w)$ and hence $H_{\Pi^*}^1(w) \leq F(w)$ for all $w \in [0, 1]$. Moreover, $H_{\Pi^*}^1(w) = 0 < F(w)$ for all $w \in (0, \Pi^*]$. Therefore

$$\int_0^1 w dH_{\Pi^*}^1(w) > \int_0^1 v dF(v) > \Pi^* = \int_0^1 w dH_{\Pi^*}^{\Pi^*}(w).$$

As $\int_0^1 w dH^B_{\Pi^*}(w)$ is continuous and strictly increasing in B, there must be a unique $B^* \in (\Pi^*, 1)$ such that $\int_0^1 w dH^{B^*}_{\Pi^*}(w) = \int_0^1 v dF(v)$. □

F being a mean-preserving spread of $H^{B^*}_{\Pi^*}$ means

$$\int_0^w F(z)dz \geq \int_0^w H^{B^*}_{\Pi^*}(z)dz \quad \text{for all } w \in [0,1], \text{ with equality for } w = 1.$$

We next show that the above inequality is strict for all $w \in (0, 1)$.

Claim 2. *$\int_0^w F(z)dz > \int_0^w H^{B^*}_{\Pi^*}(z)dz$ for all $w \in (0, 1)$.*

Proof. Define

$$\Gamma(w) := \int_0^w (F(z) - H^{B^*}_{\Pi^*}(z))dz.$$

We have to prove that $\Gamma(w) > 0$ for all $w \in (0, 1)$. For $w \in (0, \Pi^*]$, $\Gamma(w) = \int_0^w F(z)dz > 0$. For $w \in [\Pi^*, B^*)$, $F(w) - H^{B^*}_{\Pi^*}(w)$ is continuous, and so we can differentiate Γ to get

$$\Gamma'(w) = F(w) - H^{B^*}_{\Pi^*}(w) = \frac{\Pi^* - w[1 - F(w)]}{w} \geq 0,$$

where the inequality holds since $\Pi^* = \max_p [1 - F(p)]p$. Therefore, $\Gamma(w) > 0$ also for $w \in (\Pi^*, B^*)$. For $w \in [B^*, 1)$,

$$\begin{aligned}\Gamma(w) = \int_0^w (F(z) - H^{B^*}_{\Pi^*}(z))dz &= -\int_w^1 (F(z) - H^{B^*}_{\Pi^*}(z))dz \\ &= \int_w^1 (1 - F(z))dz > 0.\end{aligned}$$ □

We can now establish Claim 3, the main step of the proof.

Claim 3. *There exists $q < \Pi^*$ and $B \in [q, 1]$ such that F is a mean preserving spread of H^B_q.*

Proof. Take any $B \in (B^*, 1]$, which exists by Claim 1. For $q \leq \Pi^*$, $\int_0^1 H^B_q(w)dw$ is strictly decreasing in q. By the Dominated Convergence Theorem, $\int_0^1 H^B_q(w)dw$ is furthermore continuous in q. As $\int_0^1 H^B_0(w)dw > \int_0^1 H^{B^*}_{\Pi^*}(w)dw$ and $\int_0^1 H^B_{\Pi^*}(w)dw < \int_0^1 H^{B^*}_{\Pi^*}(w)dw$, it follows that there is a unique $q(B) < \Pi^*$ such that

$$\int_0^1 H^B_{q(B)}(w)dw = \int_0^1 H^{B^*}_{\Pi^*}(w)dw = \int_0^1 F(w)dw. \tag{A.10}$$

For every $w \in [0,1]$ and every sequence of values $B \in (B^*, 1]$, the Dominated Convergence Theorem gives

$$\lim_{B \to B^*} \int_0^w H^B_{q(B)}(z)dz = \int_0^w H^{B^*}_{\Pi^*}(z)dz,$$

noting that $\lim_{B\to B^*} q(B) = \Pi^*$. Choose $B', B'' \in (B^*, 1]$ such that $B' < B''$. For $w \in [0, B')$, we have

$$\int_0^w H^{B'}_{q(B')}(z)dz \leq \int_0^w H^{B''}_{q(B'')}(z)dz$$

since $q(B') > q(B'')$. Similarly, for $w \in [B', 1]$

$$\begin{aligned}\int_0^w (H^{B'}_{q(B')}(z) - H^{B''}_{q(B'')}(z))dz &= -\int_w^1 (H^{B'}_{q(B')}(z) - H^{B''}_{q(B'')}(z))dz \\ &= \int_w^1 (H^{B''}_{q(B'')}(z) - 1)dz \leq 0.\end{aligned}$$

By Dini's Theorem, the convergence is thus uniform across w.

Claim 2 and the uniform convergence imply that there exists $\widehat{B} \in (B^*, 1]$ such that

$$\int_0^w H^{\widehat{B}}_{q(\widehat{B})}(z)dz - \int_0^w H^{B^*}_{\Pi^*}(z)dz < \int_0^w F(z)dz - \int_0^w H^{B^*}_{\Pi^*}(z)dz \quad \forall w \in (0,1).$$

By (A.10), F is thus a mean-preserving spread of $H^{\widehat{B}}_{q(\widehat{B})}$. □

Recall that p^{RS} is the smallest price q for which there exists $B \in [q, 1]$ such that F is a mean-preserving spread of H^B_q. Hence, Claim 3 implies $p^{RS} < \Pi^*$. □

Proof of Proposition 7. According to Roesler and Szentes (2017, Lemma 1), there exists a unique B^* such that F is a mean-preserving spread of $H^{B^*}_{\Pi^*}$. The information structures in the RS class that induce buyer payoff $\overline{U}$ are thus all $(S, (G_v))$ that induce the CDF of posterior valuations $H^{B^*}_{\Pi^*}$. Consider any such $(S, (G_v))$. We will show that $\int_{\{s \in S: F_s(p^*) \in (0,1)\}} d\bar{G}(s) > 0$. By Lemma 7, this implies that $(S, (G_v))$ is not weakly robust.

By contradiction, suppose $\int_{\{s \in S: F_s(p^*) \in (0,1)\}} d\bar{G}(s) = 0$. Then,

$$\begin{aligned}\int_{p^*}^1 w dH^{B^*}_{\Pi^*}(w) = \int_{\{s \in S: F_s(p^*)=0\}} \int_0^1 v dF_s(v) d\bar{G}(s) &= \int_S \int_{p^*}^1 v dF_s(v) d\bar{G}(s) \\ &= \int_{p^*}^1 v dF(v), \qquad \text{(A.11)}\end{aligned}$$

where the last equality follows from the definition of the posterior in (1). We consider two cases. **Case 1:** $B^* \leq p^*$. As $H^{B^*}_{\Pi^*}(w) = 1$ for all $w \geq B^*$, we have a contradiction to

(A.11). **Case 2:** $B^* > p^*$. By the definition of the RS class, $[1 - H_{\Pi^*}^{B^*}(p)]p = \Pi^*$ for all $p \in [\Pi^*, B^*]$. On the other hand, $[1-F(p)]p < \Pi^*$ for all $p < p^*$. Consequently, $H_{\Pi^*}^{B^*}(p) < F(p)$ for all $p \in (0, p^*)$, whereas $H_{\Pi^*}^{B^*}(p^*) = F(p^*)$. We thus have $\int_0^{p^*} w dH_{\Pi^*}^{B^*}(w) > \int_0^{p^*} v dF(v)$. Given (A.11), this implies that F is not a mean-preserving spread of $H_{\Pi^*}^{B^*}$; a contradiction. □

Proof of Lemma 8. The proof of Lemma 1 shows that every information structure that induces price p can be extended to a p-pairwise information structure such that buyer and seller payoff at price p remain unchanged. Under every p-pairwise information structure, $H(q) \geq F(q)$ for all $q > p$. Hence, $[1 - H(q)]q \leq [1 - F(q)]q \leq \Pi^*$ for all $q > p$, and so the extended information structure does not induce any price $q > p$. If it induces a price $q < p$, then the original information structure was not buyer robust. □

Proof of Proposition 8. In the main text. □

References

Bergemann, D., B. Brooks, and S. Morris (2015): "The Limits of Price Discrimination," *American Economic Review*, 105(3), 921–957.

Bergemann, D., and M. Pesendorfer (2007): "Information Structures in Optimal Auctions," *Journal of Economic Theory*, 137, 580–609.

Board, S., and J. Lu (2017): "Competitive Information Disclosure in Search Markets," *Journal of Political Economy*, forthcoming.

Dudley, R. (2002): *Real Analysis and Probability*. Cambridge University Press, Cambridge.

Eső, P., and B. Szentes (2007): "Optimal Information Disclosure in Auctions and the Handicap Auction," *Review of Economic Studies*, 74(3), 705–731.

Garcia, D., R. Teper, and M. Tsur (2018): "Information Design in Insurance Markets: Selling Peaches in a Market for Lemons," mimeo.

Gentzkow, M., and E. Kamenica (2017): "Competition in Persuasion," *Review of Economic Studies*, 84(1), 300–322.

Grossman, S. J., and O. D. Hart (1980): "Disclosure Laws and Takeover Bids," *Journal of Finance*, 35(2), 323–340.

Guo, Y., and E. Shmaya (2017): "The Interval Structure of Optimal Disclosure," mimeo.

Kamenica, E., and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101(6), 2590–2615.

Kessler, A. (1998): "The Value of Ignorance," *Rand Journal of Economics*, 29(2), 339–354.

Koessler, F., and R. Renault (2012): "When Does a Firm Disclose Product Information?," *Rand Journal of Economics*, 43(4), 630–649.

Kolotilin, A., M. Li, T. Mylovanov, and A. Zapechelnyuk (2017): "Persuasion of a Privately Informed Receiver," *Econometrica*, 85(6), 1949–1964.

Lewis, T. R., and D. E. M. Sappington (1994): "Supplying Information to Facilitate Price Discrimination," *International Economic Review*, 35(2), 309–327.

Li, F., and P. Norman (2017): "Sequential Persuasion," mimeo.

Li, H., and X. Shi (2017): "Discriminatory Information Disclosure," *American Economic Review*, 107(11), 3363–3385.

Marshall, A. W., I. Olkin, and B. C. Arnold (2011): *Inequalities: Theory of Majorization and Its Applications*, Springer Series in Statistics. Springer New York.

Milgrom, P. (1981): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12(2), 380–391.

Nelsen, R. B. (2006): *An Introduction to Copulas*, Springer Series in Statistics. Springer New York.

Nikandrova, A., and R. Pancs (2017): "Conjugate Information Disclosure in an Auction with Learning," *Journal of Economic Theory*, 171, 174–212.

Perez-Richet, E., and V. Skreta (2017): "Information Design under Falsification," mimeo.

Roesler, A.-K., and B. Szentes (2017): "Buyer-Optimal Learning and Monopoly Pricing," *American Economic Review*, 107(7), 2072–2080.

Rudin, W. (1987): *Real and Complex Analysis*. McGraw-Hill, New York, third edn.

Shaked, M., and J. G. Shanthikumar (2007): *Stochastic Orders*, Springer Series in Statistics. Springer New York.

von Wangenheim, J. (2017): "Consumer-Optimal Information Design," mimeo.