

# SUPPLEMENT TO “MAXIMALITY IN THE FARSIGHTED STABLE SET”

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## ONLINE APPENDIX

### A.1. COMPETITIVE EQUILIBRIA AND ABSOLUTE MAXIMALITY

As noted in Section 3.2, several classes of games are known to have separable allocations. Each such payoff allocation corresponds to an absolutely maximal farsighted stable set. Remark 2 identifies another important case: a competitive equilibrium when preferences satisfy local non-satiation and strict convexity.

**Proof of Remark 2.** Consider a competitive equilibrium  $(\{\xi_i\}, p)$ . We claim that  $u \equiv \{u_i(\xi_i)\}$  is separable. Suppose there is a coalition  $S$  such that  $u_S \in V(S)$ . This means that there exists a feasible allocation  $\xi'$  for the economy with agent set  $S$  such that  $\sum_{i \in S} \xi'_i = \sum_{i \in S} \omega_i$  and  $u_i(\xi'_i) \geq u_i$  for all  $i \in S$ . Since preferences are locally non-satiated, condition (i) of a competitive equilibrium implies that  $p \cdot \xi'_i \geq p \cdot \omega_i$  for all  $i \in S$ . In fact, it must be the case that

$$(1) \quad p \cdot \xi'_i = p \cdot \omega_i \text{ for all } i \in S,$$

otherwise we contradict the feasibility condition  $\sum_{i \in S} \xi'_i = \sum_{i \in S} \omega_i$ . Next, we claim that  $\xi'_i = \xi_i$  for all  $i \in S$ . If not, there is some  $i \in S$  with  $\xi'_i \neq \xi_i$ . By the strict convexity of  $u_i$ , there is a strict convex combination of  $\xi'_i$  and  $\xi_i$  which is strictly preferred to  $\xi_i$ . By (1) it is also affordable. But this contradicts condition (i) of a competitive equilibrium. It follows that  $\sum_{i \in S} \xi_i = \sum_{i \in S} \xi'_i = \sum_{i \in S} \omega_i$ . Because  $\sum_{i \in N} \xi_i = \sum_{i \in N} \omega_i$ , this implies that  $\sum_{j \in N-S} \xi_j = \sum_{j \in N-S} \omega_j$ , and  $u_{N-S} \in V(N-S)$ ; i.e.,  $u$  is separable. ■

Although Remark 2 applies even to economies in which the interior of the core is empty, it does depend crucially on preferences being *strictly* convex. Consider an exchange economy with three consumers and two commodities that are perfect complements:  $u_i(x_i) = \min\{x_{i1}, x_{i2}\}$  for all  $i$ ; preferences are convex but not *strictly* convex. If the endowment of consumer 1 is  $(1, 0)$  and the endowments of the other two are  $(0, 1)$ , the unique competitive payoff is  $(1, 0, 0)$ . This is *not* separable because agents 2 and 3 can achieve this on their own but agent 1 cannot get 1 on her own.<sup>1</sup>

Greenberg, Luo, Oladi and Shitovitz (2002) study what they call “the sophisticated stable set” of an exchange economy. This is based on a version of the Harsanyi stable set in which every step of a blocking chain is also required to be a myopic objection. The core is consequently a subset of the sophisticated stable set. In general, therefore, the

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<sup>1</sup>The coalitional game in this case is the same as the veto game of Example 4 in Appendix A.5.

competitive equilibrium is not a single-payoff sophisticated stable set. A second difference between the farsighted stable set and the sophisticated stable set is that the latter, as in Harsanyi (1974), allows a deviating coalition to choose *any* feasible payoff for the complementary coalition, a notion that is critiqued and dropped in Ray and Vohra’s (2015) development of the farsighted stable set. Last but not least, our focus here is on maximality.

## A.2. ABSOLUTE MAXIMALITY IN SIMPLE GAMES

Simple games have proved to be fertile ground for studying stable sets as well as farsighted stable sets. In this section, we prove Remark 3 in the main text.

We identify all configurations with no winning coalitions with a single state, and call this the *zero state*. The first part of Remark 3 is a consequence of farsighted internal stability. To prove it, suppose there is a farsighted stable set  $F$  for which Property B fails. Then there are states  $a$  and  $b$  in  $F$  and a coalition  $T$  — it must be winning — such that  $u_T(b) \gg u_T(a)$  and  $\sum_{i \in T} u_i(b) \leq 1$ . Because  $T$  is winning, its complement is losing. So at state  $a$ ,  $T$  can precipitate the zero state (by breaking up into singletons), counting on the winning coalition for state  $b$  to move to  $b$ , making  $T$  better off. Therefore  $b$  farsightedly dominates  $a$ , which contradicts the farsighted internal stability of  $F$ . So Property B must hold.

To proceed further, recall that a *veto player* in a simple game is an individual with a losing complement (she can single-handedly precipitate the zero state). If the set of all veto players is winning, say that the game is oligarchic. Oligarchic games have singleton farsighted stable sets (Ray and Vohra 2015, Theorem 3).

To examine larger stable sets in non-oligarchic games, consider *discriminatory sets*:

$$D(K, c) = \{x \in X \mid u_i(x) = c_i \text{ for } i \in K\}$$

for some fixed player set  $K \subseteq N$  and associated payoff vector  $c \in \mathbb{R}_{++}^K$ . Those in  $K$ , the “discriminated players,” each get a fixed (positive) amount, while the remaining surplus is divided arbitrarily among the remainder, the “bargaining players.” As shown in Ray and Vohra (2015, Theorem 5), such sets exist in every standard simple game.<sup>2</sup> The following Remark proves Remark 3:

REMARK 4. Every discriminatory farsighted stable set satisfies Property A as well as Property A’.

*Proof.* Let  $a, b \in D(K, c)$ , with  $u_j(b) > u_j(a)$  for some  $j$ . Clearly,  $j \notin K$ , which means that there is  $z \in D(K, c)$  with  $u_k(z) = c_k$  for all  $k \in K$ ,  $u_i(z) > u_i(b)$  for all

<sup>2</sup>Note that they use the term *nonelitist* (rather than “standard”) to refer to a veto coalition with replaceable members.

$i \in N - K - j$ , and  $u_j(z)u_j(a)$ . Therefore  $D(K, c)$  satisfies Property A. If  $b$  is a regular state, then so is  $z$ , which means that  $D(K, c)$  also satisfies Property A'. ■

Recall that in Example 1 there is a farsighted stable set which is not absolutely maximal. That game is a regular non-oligarchic simple game; coalition  $\{1, 2, 3\}$  is a veto coalition with replaceable members. So it has a discriminatory farsighted stable set, e.g.,  $D(\{4\}, 0.1)$ , which is absolutely maximal.

For simple games, another well-studied set — with a discrete collection of payoffs — is a potential candidate for a stable set (von Neumann and Morgenstern 1944). For any vector  $m \in \mathbb{R}^N$  with  $m \gg 0$  and  $\sum_{i \in S} m_i = 1$  for every minimal winning coalition  $S$ , define

$$Z(m) = \{x \in X \mid S(x) \text{ is minimal winning and } u_i(x) = m_i \text{ for } i \in S\}$$

to be a *main simple set*. von Neumann and Morgenstern (1944) showed that if a game is *strong* — every coalition is either winning, or its complement is — then the set of utility profiles corresponding to a main simple set is a vNM stable set. Ray and Vohra (2015) showed that a main simple set (of a strong, simple game) is a farsighted stable set.

In general, a main simple set may not satisfy Property A, as we saw in Example 1. But an important subclass of simple games yields a different answer. Say that a simple game is *symmetric* if there is some  $k$ , where  $(n+1)/2 \leq k \leq n$ , such that every coalition with  $k$  players is a minimal winning coalition. (Supermajority games have this property.) Every symmetric simple game has a main simple set  $Z(m)$ , with  $m_i = 1/k$  for all  $i$ . Observe that a symmetric game may not be strong. Yet its main simple set is indeed a farsighted stable set, though it may fail to be a vNM stable set.<sup>3</sup>

Moreover, if the game is non-oligarchic ( $k < n$ ),  $Z(m)$  satisfies Property A. To see this, suppose  $a$  and  $b$  are in  $Z(m)$ , with  $u_i(b) > u_i(a)$ . This implies that there is a minimal winning coalition  $S$  such that  $i \in S$ ,  $u_i(b) = 1/k$ , and  $u_j(a) = 0$  for all  $j \notin S$ . Since  $k < n$ , there exists  $j \notin S$ . Let  $S' = S - i + j$ . Given the symmetry of the game, this is a

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<sup>3</sup>Consider the non-strong, symmetric simple game with  $n = 5$  and  $k = 4$ . Then  $Z(m)$  is not vNM stable: the state  $x$  with  $u(x) = (1/3, 1/3, 1/3, 0, 0)$  has no objection from  $Z(m)$ . However, there is a farsighted objection through the zero state initiated by players 4 and 5 (a veto coalition), leading to  $Z(m)$ . Indeed, farsighted stability holds for all such games. To see why  $Z(m)$  satisfies farsighted external stability, consider  $x \notin Z(m)$ . Clearly,  $S = \{i \in N \mid u_i(x) \geq 1/k\}$  must then be a losing coalition. If the complement of  $S$ ,  $N - S = \{i \in N \mid u_i(x) < 1/k\}$ , is winning, any minimal winning coalition in  $N - S$  can (myopically) block  $x$  with a state in  $Z_m$ . Otherwise, because  $S$  is losing,  $N - S$  is a veto coalition and can farsightedly block  $x$  by first precipitating the zero state and then moving (via a suitable minimal winning coalition) to obtain  $1/k$  for all its members. The main simple set also satisfies farsighted internal stability (under a mild monotonicity restriction on the effectivity correspondence); see Ray and Vohra (2015) or Dutta and Vohra (2017). Thus  $Z(m)$  is a farsighted stable set. These arguments can be extended to show that a main simple set of any (not necessarily symmetric) simple game is a farsighted stable set, although absolute maximality cannot be assured, as shown by Example 1.

minimal winning coalition and the corresponding state in  $Z(m)$ , say  $z$ , has the property that  $u_i(z) = 0$  and  $u_j(z) \geq u_j(b)$  for all  $j \neq i$ , which yields Property A. Because  $S'$  is a minimal winning coalition,  $z$  is a regular state, and so Property A' is also satisfied. To summarize:

REMARK 5. Any non-oligarchic, symmetric simple game possesses a main simple set which is a farsighted stable set and satisfies Property A as well as Property A'.

### A.3. MORE REMARKS ON COALITIONAL ACCEPTABILITY

In Section 4 of the paper, we discuss a parallel to Theorem 1 to blocking processes; that is, to ambient processes that employ blocking chains following *every* history. The resulting Proposition 1 is, however, quite restrictive and does not apply to all characteristic functions. We re-do that Proposition here by imposing a still stronger version of Property A. Recall that  $\sigma$  is a *blocking process* if for each history  $h$ , if  $S(h)$  is nonempty, then  $u_{S(h)}(x^\sigma(h)) > u_{S(h)}(x(h))$ .

Recall also the discussion from the main text. To extend Theorem 1 to blocking processes, it suffices to strengthen Lemma 2 so that the coalitionally acceptable chain constructed to deter deviations is in fact a blocking chain. At a minimum, this will require that when we dissuade an off-path deviation by finding a coalitionally acceptable chain from  $y$  to  $z \in F$ , all players involved in this chain must receive a strictly positive payoff at  $z$ . That motivated a modification of our Property A to regular states; see Property A' in the main text. We go still further here by allowing for strict inequalities.

**Property A''.** *Suppose there are two regular states  $a$  and  $b$  in  $F$  such that  $u_j(b) > u_j(a)$  for some  $j$ . Then there exists a regular state  $z \in F$  such that  $u_j(z) \leq u_j(a)$ , and  $u_i(z) \geq u_i(b)$  for all  $i \neq j$ , with strict inequality holding for player  $i \neq j$  if and only if  $i$ 's coalition in state  $z$  is different from her coalition in state  $b$ .*

**THEOREM 2.** *If a farsighted stable set satisfies Properties A'' and B, then it can be embedded in an absorbing, absolutely maximal blocking process.*

Observe that a single-payoff farsighted stable set trivially satisfies Property A''.

We leave it to the reader to check that to prove Theorem 2 it suffices to prove the following version of Lemma 2 in which the conclusion relates to a blocking process rather than a coalitionally acceptable process.

**LEMMA 3.** *Consider a farsighted stable set  $F$  that satisfies Properties A'' and B. Suppose  $T$  moves from state  $x \notin F$  to state  $y$ ,  $\Psi(x) = a$  and  $\Psi(y) = b$ . Then there is a state  $z \in F$  and a blocking chain from  $y$  to  $z$  such that  $u_j(z) \leq u_j(a)$  for some  $j \in T$ .*

*Proof.* Consider states  $x, y, a, b$  and a coalition  $T$  as in the statement of Lemma 3. If the conclusion of the Lemma is false,  $u_T(b) \gg u_T(a)$ . As in the proof of Lemma 2, Condition B implies that  $y \neq b$ , so that  $y \notin F$ . Moreover, as in that proof, we have a player  $j \in T$  and a blocking chain  $\mathbf{c}' = \{y, y^1, \dots, y^{m-1}, y', y^m\}, \{S^1, \dots, S^{m-1}, W - j, S^m\}$  where  $y^m = b$  and  $S^m = \cup_{j=1}^{m-1} S^j \cup W$  is the set of all players who actively move in the blocking chain  $\mathbf{c}'$ . By Condition A'', there is a regular state  $z$  such that  $u_j(z) \leq u_j(b)$  and  $u_i(z) \geq u_i(b)$  for all  $i \neq j$ , with strict inequality holding for player  $i \neq j$  if and only if  $i$ 's coalition in state  $z$  is different from her coalition in state  $b$ . Modify  $\mathbf{c}'$  by replacing the terminal state with  $z$  to construct the chain  $\bar{\mathbf{c}} = \{y, y^1, \dots, y^{m-1}, y', z\}, \{S^1, \dots, S^{m-1}, W - j, S'\}$ , where  $S'$  is a minimal set of players needed to replace  $y'$  with  $z$ . We will show that  $\bar{\mathbf{c}}$  is a blocking chain. Recall from the proof of the Claim that there can be only two possible reasons why  $\bar{\mathbf{c}}$  may be a coalitionally acceptable but not a blocking chain:

- (a) there exists a player  $k \in S' - S^m$  for whom  $u_k(z) = u_k(y') = u_k(b)$ , or
- (b)  $j \in S'$  and  $u_j(z) = u_j(y') = 0$ .

To complete the proof we will rule out each of these possibilities.

Suppose (a) holds and  $k \in S' - S^m$  is such that  $u_k(z) = u_k(y') = u_k(b)$ . Let  $S_k \in \pi(y')$  be the coalition that contains player  $k$ . Given the construction of  $z$ , this must mean that  $k$  continues to belong to coalition  $S_k$  in  $\pi(z)$ , even though  $k$ , being a member of  $S'$ , is an active mover from  $y'$  to  $z$ . In fact, by Property A'',  $u_i(z) = u_i(y')$  for all  $i \in S_k$ . But this contradicts the fact that  $S'$  is a minimal set of players needed to replace  $y'$  with  $z$ .

Suppose  $j \in S'$ . Since  $z$  is a regular state,  $u_j(z) > 0$ , so (b) cannot hold. ■

#### A.4. OTHER APPROACHES TO MAXIMAL FARSIGHTEDNESS

We seek conditions under which a farsighted stable set might satisfy maximality. The underlying idea is to begin with a solution concept that is the natural farsighted extension to a classical notion — vNM stability — and attempt to embed that concept within an ambient negotiation process satisfying the desideratum of absolute maximality. Alternatively, one might directly seek to understand the absorbing states of a negotiation process, without asking that any existing solution concept be embedded in it. Such an approach is followed in Konishi and Ray (2003) and Dutta and Vohra (2017), along with the additional restriction that the negotiation process  $\sigma$  is Markovian or history independent: for any two histories  $h$  and  $h'$ ,  $x(h) = x(h')$  implies that  $\sigma(h) = \sigma(h')$ .<sup>4</sup> A comparison of these two approaches is instructive.

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<sup>4</sup>History-dependent versions of these solutions are studied in Hyndman and Ray (2007), Ray and Vohra (2014) and Dutta and Vartiainen (2018).

1. While unclear from its original definition (Harsanyi 1974, Ray and Vohra 2015), a farsighted stable set is fundamentally a history-dependent object. There is little hope of being able to embed a farsighted stable set in a Markovian process, and this is so even if we ignore the maximality requirement. On the other hand, as our main results demonstrate, permitting history dependence can make it possible to embed a farsighted stable set in a process that is *absolutely maximal*.

2. Absolute maximality can be a more stringent requirement than maximality or strong maximality even if the focus is on absorbing states that are not necessarily a farsighted stable set.<sup>5</sup>

EXAMPLE 4. A three-player veto game:  $N = \{1, 2, 3\}$ ,  $\nu(N) = \nu\{1, 2\} = \nu\{1, 3\} = 1$  and  $\nu(S) = 0$  for all other  $S$ .

Ray and Vohra (2015) show that *every* farsighted stable set of this game is a discriminatory set of the form  $Z_a = D(\{1\}, a)$  in which player 1 receives a fixed payoff  $a \in (0, 1)$  and the remaining surplus is divided in any arbitrary way between players 2 and 3. (In fact, for every  $a \in (0, 1)$ ,  $Z_a$  is a farsighted stable set.) By Remark 4 it is absolutely maximal.

But this result depends crucially on allowing the process to be history dependent. As Dutta and Vohra (2017) point out, a set of this form cannot be supported by a Markovian process that is consistent with farsighted external stability. The farsighted external stability of  $Z_a$  implies that from any state  $x$  with  $u(x) \gg 0$  and  $u_1(x) > a$  there is a blocking chain ending in  $Z_a$ . It can be shown that any such chain involves players 2 and 3 leaving the grand coalition at state  $x$ , resulting in the zero state. This is then followed by a move by  $N$  to a state in  $Z_a$ ; see Ray and Vohra (2015) for details. It turns out that the last step of any such blocking chain *must* depend on the history.

To see this, suppose  $\sigma$  is a Markovian process that defines, for every state not in  $Z_a$ , a blocking chain that ends in  $Z_a$ . Consider the zero state,  $x^0$ , and suppose  $\sigma$  prescribes a path from  $x^0$  that ends at  $y \in Z_a$ . Since the process is Markovian, this is the continuation paths for *all* histories where the current state is  $x^0$ . Consider  $x$  such that  $u(x) \gg 0$  and  $u_1(x) > a$ . As already observed, any blocking chain from  $x$  leading into  $Z_a$  must involve coalition  $\{2, 3\}$  moving to  $x^0$ , followed by a move by  $N$  to  $y$  (with  $u(y') \gg 0$ ). Since  $u(y') \gg 0$ , we can find  $x$  such that  $u_1(x) > a$ ,  $u_2(x) > u_2(y)$  and  $u_3(x) > 0$ . The process must specify a blocking chain from  $x$  to a state in  $Z_a$ . But any such blocking chain must be one in which  $\{2, 3\}$  first moves to  $x^0$  followed by a move to  $y$ . Since  $u_2(x) > u_2(y)$  player 2 cannot gain. In other words, the path prescribed by  $\sigma$  from  $x$  is not a blocking chain, a contradiction.

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<sup>5</sup>With respect to farsighted stable sets this point has already made; through Example 1 for simple games and through the examples in Dutta and Vohra (2017) for abstract games.

This example also illustrates the difference between our approach and one that directly examines the absorbing states of a process, without seeking to embed a particular solution. Consider the Dutta and Vohra (2017) notion of a *strong rational expectations farsighted stable set* (SREFS) which is defined to be the set of absorbing states,  $Z$ , of a Markovian process  $\sigma$  that satisfies strong maximality, as well as both internal and external stability when blocking chains are restricted to be consistent with  $\sigma$ . In particular, if a coalition moves from a state in  $Z$ , it cannot eventually gain *provided* the continuation following this move is given by  $\sigma$ . There may, however, exist a farsighted blocking chain that is not consistent with  $\sigma$ , and for this reason  $Z$  may not be a farsighted stable set.<sup>6</sup> Indeed this is a feature of the present example. Dutta and Vohra (2017) show that there is a SREFS consisting of states with payoffs  $(\{a + b, b, 0\}, (a + b, 0, b), (a, b, b)\})$ , where  $a \in (0, 1)$  and  $b = (1 - a)/2$ . Of course, this is not a discriminatory set in which player 1 gets a fixed payoff, so it cannot be a farsighted stable set.<sup>7</sup>

While this SREFS satisfies strong maximality, it does not satisfy absolute maximality. To see this, consider the state  $x$  with  $\pi(x) = N$  and  $u(x) = (a + b - 1/3\epsilon, b - 2/3\epsilon, \epsilon)$ . Coalition  $\{1, 2\}$  can block this in one step to get payoffs  $(a + b, b)$ . No coalition that includes either player 1 or 2 can construct a better deviation, as is required for strong maximality. However, absolute maximality may not hold because of a deviation by player 3. Suppose that a departure by player 3 results in the other two sharing the extra surplus equally. Now, if player 3 leaves the grand coalition, the new state leaves player 2 with a payoff strictly less than  $b$ . This only leads to the zero state followed by  $N$  moving to the stationary state with payoffs  $(a, b, b)$ . Thus, player 3, has a profitable deviation at state  $x$ , and the process is not absolutely maximal.

We make a final comment on folk-theorem-like arguments. In Section 3.4, we remarked that there are tight restrictions on the structure of absolutely maximal farsighted stable sets, so it isn't the case that anything goes. That argument carries over to the set of states that comprise any farsighted stable set: "anything doesn't go" because the internal stability of a (farsighted) stable set precludes it from being too inclusionary. However, what would happen under the alternative approach of this section, where we do not insist in embedding any farsighted stable set? Might that span the entire set of feasible payoffs? In general, in our model, the answer is still no. That follows from absolute maximality and our notion of an absorbing state which requires once such a state is reached, regardless of the history, it does not change. Together, these two properties imply that a non-core state and a state that (myopically) dominates it can't both be

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<sup>6</sup>The same is true of the solutions constructed by Dutta and Vartiainen (2018), allowing for history dependence, and using a weaker notion of absorption that we have defined above. Indeed, their solutions may not even satisfy myopic internal stability.

<sup>7</sup>There is a farsighted objection from  $((a, b, b), N)$ , led by player 1, to  $((a + b, b, 0), \{1, 2\})$ .

absorbing states.<sup>8</sup> Thus, in general, the absorbing states cannot span the entire set of feasible payoffs.

A loosening of these restrictions could lead to outcomes in which almost the entire set of feasible payoffs is supportable. Under the weaker notion of absorption used by Dutta and Vartiainen (2018), “stable states” need not satisfy internal stability even if the process is maximal. In fact, in a strictly superadditive game, they coincide with the set of all strictly positive, feasible payoffs.

#### A.5. PROPERTIES OF EXAMPLES 1, 2 AND 3

##### Properties of Example 1

To show that  $F$  is not absolutely maximal, we impose a “monotonicity condition” on the effectivity correspondence  $E$ .<sup>9</sup> Assume that if a winning coalition loses some members but remains winning, the resulting nonnegative surplus (captured from the departing members) is shared equally among the players that remain. Now suppose by way of contradiction that there is an absorbing  $\sigma$  that embeds  $F$  and satisfies coalitional acceptability and absolute maximality. Consider state  $x$  with  $u(x) = (0, 0, 0.36, 0.64)$  and winning coalition  $W(x) = N$ . Because  $x \notin F$ , there is  $x' \in F$  that farsightedly dominates it; i.e.,  $\sigma$  leads from history  $h = \{x\}$  to  $x'$ . Ray and Vohra (2015, Lemma 2) show that there are just two possibilities: either (i)  $x'$  myopically dominates  $x$ , or (ii)  $W^+ = \{i \in N \mid u_i(x') > u_i(x)\}$  and  $W(x) - W^+$  are both losing coalitions.<sup>10</sup> But  $W(x)$  equals  $N$  and our game is strong, so the second option must be eliminated here. It follows that (i) is true:  $x'$  myopically dominates  $x$ . But the only two states in  $F$  that do so are  $x' = ((1/3, 0, 0, 2/3), \{1, 4\})$  or  $x' = ((0, 1/3, 0, 2/3), \{2, 4\})$ . In either case,  $u_3(x') = 0$ . We use this last fact to argue that player 3 can profitably deviate from the stipulated move at  $x$  (to  $x'$ ), thus violating absolute maximality.

Suppose player 3 leaves the grand coalition at  $x$  resulting in state  $y$ . Note that the residual coalition,  $\{1, 2, 4\}$  is winning. Given that the residual players share equally in the surplus released by 3’s departure,  $u(y) = (0.12, 0.12, 0, 0.76)$ . Since  $y \notin F$ ,  $\sigma$  must prescribe a continuation that is coalitionally acceptable. Using the same kind of argument as in the previous paragraph, it can be shown that  $x^\sigma(y) = ((1/3, 1/3, 1/3, 0), \{1, 2, 3\})$ .<sup>11</sup> Player 3 can therefore gain by interfering in this way with any process that attempts to proceed from  $x$  to  $x'$ . In other words,  $F$  does not satisfy absolute maximality.

<sup>8</sup>In other words, the set of absorbing states must satisfy myopic internal stability.

<sup>9</sup>See Ray and Vohra (2015) for a more general version that applies to all games.

<sup>10</sup>In case (ii),  $W^+$  can precipitate the zero state by leaving  $W(x)$ , followed by a move by  $W(x')$  to  $x'$ .

<sup>11</sup>For example, if  $x^\sigma(y) = ((1/3, 0, 0, 2/3), \{1, 4\})$  coalitional rationality implies that in the first step player 1 must leave  $W(y)$ , resulting in the state  $y' = ((0, 0.18, 0, 0.82), \{2, 4\})$ . But from  $y'$  it is not possible, by coalitional rationality, to end up at  $((1/3, 0, 0, 2/3), \{1, 4\})$ .



## Properties of Example 2

We first show that  $F$  is a farsighted stable set. Figure 1 shows all the payoff equivalent states, with arrows indicating the states in  $F$  that farsightedly dominate a state not in  $F$ .

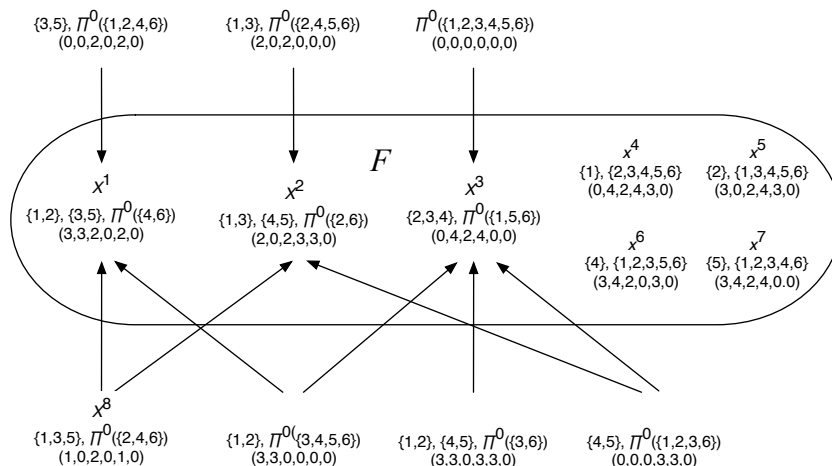


FIGURE 1. External Farsighted Stability of  $F$  in Example 2

*To see that  $F$  satisfies external farsighted stability:* A state with payoff  $(3, 3, 0, 3, 3, 0)$  is dominated by one in  $X^3$  through coalition  $\{2, 3, 4\}$ . The state with payoff  $(3, 3, 0, 0, 0, 0)$  is directly dominated by one in  $X^1$  through  $\{3, 5\}$  and by one in  $X^3$  through  $\{2, 3, 4\}$ . A state in  $X^8$ , with payoff profile  $(1, 0, 2, 0, 1, 0)$ , is farsightedly dominated by a state in  $X^1$  through the formation of coalition  $\{1, 2\}$ , and also by a state in  $X^2$  through coalition  $\{4, 5\}$ . It is easy to see from Figure 1 that other states not in  $F$  are also farsightedly dominated by some state(s) in  $F$ .

*To see that  $F$  satisfies internal farsighted stability:* First observe that states  $x^4, x^5, x^6$  and  $x^7$  cannot dominate any other state (these states are in  $F$  only because they cannot be dominated by a state in  $X^1 \cup X^2 \cup X^3$ ). This is so because such a state can emerge in only one of two ways: either a singleton precipitates it by leaving the grand coalition or it involves the active participation of player 6. Either case is inconsistent with farsighted dominance because both the singleton as well as player 6 receive 0. Secondly, none of these states can be farsightedly dominated by any other state. All players except for the excluded singleton are receiving the maximum possible payoff. Only the singleton has an incentive to change the state, but on her own she is powerless to do so. Thus, in checking internal stability we only need to compare states in  $X^1, X^2$  and  $X^3$ .

From  $X^1$  the only players who could gain by ending up at a state in  $X^2$  are players 4 and 5. They can't move there directly. They could form a coalition of their own, resulting in payoffs  $(3, 3, 0, 3, 3, 0)$ , but that can only be dominated by a state in  $X^3$ , not one in  $X^2$ ,

resulting in a payoff of 0 to player 5, which is of course not a farsighted improvement for  $\{4, 5\}$ . Player 5 could exit coalition  $\{3, 5\}$  resulting in payoffs  $(3, 3, 0, 0, 0, 0)$  but from there the only possible moves are into  $X^1$  or  $X^3$ , again making it impossible for player 5 to gain.

A state in  $X^1$  cannot be farsightedly dominated by one in  $X^3$  because any such move must begin by player 2 leaving coalition  $\{1, 2\}$  which results in payoffs  $(0, 0, 2, 0, 2, 0)$  from which the only further move that is possible is to  $X^1$  or to  $X^2$ , not  $X^3$ , because players 3 or 5 the only ones who could initiate a move to  $X^3$  have no interest in doing so. A similar argument shows that no state in  $X^2$  can be farsightedly dominated by another state in  $F$ . Finally, note that at a state in  $X^3$ , all the non-zero payoff players are getting the highest possible amount and they together belong to one coalition, so no profitable deviation is possible.

This completes the proof that  $F$  satisfies farsighted internal stability.

Finally, we show that  $F$  is not absolutely maximal; that is, any coalitionally rational and absorbing process in which it is embedded must fail to satisfy absolute maximality. Consider a state in  $X^8$  with payoffs  $(1, 0, 2, 0, 1, 0)$ . The *only* possible farsighted blocking chain from such a state ends in  $X^1$  or  $X^2$ , not in  $X^3$ . This is so because the only players who would prefer to have it replaced by one in  $X^3$  are 2 and 4, but without the active participation of player 3 they are unable to carry out such a move. Suppose  $F$  is embedded in a coalitionally acceptable and absorbing process. Consider the history consisting of a single state in  $X^8$ . Since the only blocking chains from such a state are into  $X^1$  or  $X^2$ , the continuation must be a single step into  $X^1$  (through coalition  $\{1, 2\}$ ) or into  $X^2$  (through coalition  $\{4, 5\}$ ). In the former case, coalition  $\{4, 5\}$  has a profitable deviation into  $X^2$  and in the latter coalition  $\{1, 2\}$  has a profitable deviation into  $X^1$ . Thus,  $F$  is not absolutely maximal, which also shows that Property B cannot be dispensed with in our main theorem.

### Properties of Example 3

The farsighted stability of  $F$  follows from arguments we already provided in the discussion of Example 2. In Figure 2 the arrows from states outside  $F$  represent a process that embeds  $F$ . We leave it to the reader to check that it satisfies absolute maximality.

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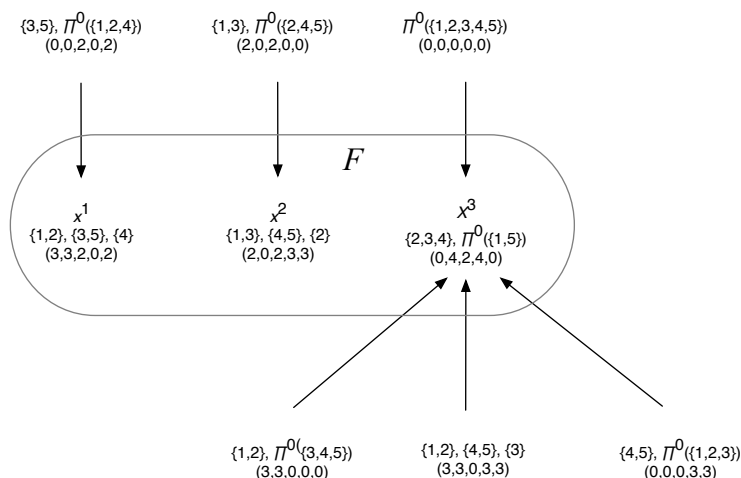


FIGURE 2. Absolute Farsighted Stability of  $F$  in Example 3

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