Coordination on Networks

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Abstract

We study a coordination game among agents on a network, who choose whether or not to take an action that yields value increasing in the actions of neighbors. In a standard global game setting, players receive noisy information of the technology's common state-dependent value. At the noiseless limit, equilibrium strategies are threshold strategies: each agent adopts if the signal received is above a certain cutoff value. We characterize properties of the cutoffs as a function of the network structure. This characterization allows to partition players into coordination sets, i.e., sets of players where all members take a common cutoff strategy and are path connected. We also show that there is a single coordination set (all players use the same strategies, so they perfectly coordinate) if and only if the network is balanced, i.e., the average degree of each subnetwork is no larger than the average degree of the network. Comparative statics exercises as well as welfare properties are investigated. We show that, in order to maximize aggregate welfare or adoption, the planner needs to target coordination sets and not individuals.

Keywords: global games, coordination, network partition, welfare.
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1 Introduction

Settings with binary actions and positive network effects are ubiquitous: the choice to adopt a technology or platform, such as in social media, where the value of the technology/platform is increasing in the adoption by friends; the choice to partake in crime when the proficiency of crime, and thus the likelihood of not getting caught, is increasing in the criminality of accomplices; or, in immigration policy when the influx of migrants depends on anti-immigration policies employed in neighboring countries. In each of these examples, uncertainty in a common state can also influence the value of adoption: the underlining value of the technology; the strength or presence of the police force; the state of the economy.

In a network setting, this paper studies coordination in these uncertain environments. We show that we can partition players into coordination sets of players so that all members of these sets have the same decision strategy and are path connected. We also show if the network is balanced (i.e., the average degree of each subnetwork is no larger than the average degree of the network), then all players in a network have the same decision strategy: they will all adopt the technology or nobody will, they will all partake in crime or nobody will, etc. Networks that are symmetric (such as the circle or the complete network), but also many networks that are asymmetric, such as stars, satisfy this property.

The ensuing model employs the tools of global games embedded into a network game. Players' positions in the network define their preferences over the action choices of others. Using the language of technology adoption, the total value an agent receives from adopting the technology is increasing in the technology's underlining value (the state) and in the adoption by neighbors. Agents receive noisy signals informing them of the state. In equilibrium, agents further use their private information to infer the observations and actions of neighboring agents, and anticipate the ultimate value they will enjoy from adopting the technology.

The classic equilibrium selection of global games obtains. In our setting with binary actions, the equilibrium selected in the noiseless limit comes in the form of cutoff strategies. Each agent adopts the technology when their private signal exceeds their equilibrium cutoff, a cutoff determined by the agent's position in the network. We explore the role of the network's architecture in determining who coordinates their adoption choices with whom. The analysis begins by calculating limiting cutoffs, which allocate players to coordination sets. These coordination sets are defined by path-connected agents taking a common cutoff strategy. The analysis then provides necessary and sufficient conditions for all agents to inhabit a common coordination set. To establish this characterization, we construct a network flow problem, and apply Gale's Demand Theorem (see Gale (1957 [23])) to establish a balance in incentives to adopt across all members of the coordination set.[1]

To study the role of network structure, we first consider the case of homogeneous values where the network alone introduces (ex ante) heterogeneity across agents. We give an exact condition for which a single coordination set exists in the network. This condition says that the network structure must be *balanced*, that is, that the average degree of each sub-network (composed of any nonempty subset of agents in the network) is not greater than the average degree of the original network. To understand this, consider any core-periphery network, with regular core of degree d_c and size n_c, and with n_p periphery nodes, each connected to k core nodes symmetrically. This graph is balanced if and only if $d_c \leq 2k$, which means that either the core is not very dense (as, for example, in a star network), or the number of links to the core is very large. Otherwise the periphery agents will have a strictly higher cutoff than the core agents and there will be more than one coordination set.

This network characterization to global coordination implies that under homogeneous values, remarkably, agents with arbitrarily different degrees may belong to the same coordination set. For example, in a star network, regardless of the number of peripheral agents, all agents coordinate together in the limit, meaning that they adopt the technology under the same set of states.[2] To better understand this result, near the noiseless limit equilibrium cutoffs for the center and the peripheral agents must lie within each others' noise supports. Therefore, in the limit, the center and periphery agents must take identical cutoff strategies. Upon increasing the size of the core, network effects within the core become sufficiently reinforcing so that the core agents may take a strictly lower cutoff than the peripheral agents. In fact, the selection of an equilibrium that exhibits a common cutoff across all agents within the network is shown to extend to all regular networks, tree networks and regular-bipartite networks.[3] For regular networks, the homogeneity of degree leads to a common cutoff. In trees, it is the absence of multiple cycles (closed walks) which guarantees a common cutoff. In regular-bipartite networks,

[1]Online Appendix D.3 generalizes the characterization to describe common coordination among subsets of agents. The intuition of balanced incentives to adopt generalizes.

[2]Indeed, using the notations above, for the star network, $d_c = 0 < 2k = 2$, so this network is balanced.

[3]A formal definition to regular-bipartite networks is provided in Section 6. A common cutoff is also shown to obtain in networks that have at most one cycle (e.g. trees) and those that have at most four agents.

the short side of the network modulates the cutoff shared by both sides.

The robust coordination exhibited in the model is no less interesting when multiple cutoffs obtain. As an initial illustration of this, we provide conditions under which additional links between coordination sets impose zero influence on the equilibrium play of the coordination set taking a lower cutoff. For example, as a lone peripheral agent sequentially links to a clique, each link influences the lone agent's cutoff while the clique remains unaffected, until a threshold number of links are established, after which the full network begins to coordinate together. We explore these properties in help-decision networks in rural India and in friendship networks in the U.S.

Upon introducing heterogeneous values, we extend and more broadly characterize the robust coordination described under homogeneous values. First, the attainment of a common cutoff within each coordination set is shown to be robust to perturbations to the intrinsic (state independent) value of the technology to any given agent. Holding fixed other parameters of the model, we characterize the range of intrinsic values that support an agent's coordination with her coordination set. The size of this support is shown to be strictly increasing in the relative size of network effects: coordination becomes increasingly *sticky* as network effects strengthen. Perturbations are shown to influence equilibrium adoption only across members within the perturbed agent's coordination set. Thus, the contagion of such perturbations extends within coordination sets, but discontinuously drops to zero across coordination sets: contagion is *contained*. On the contrary, large changes to intrinsic values can alter the composition of coordination sets across the network.

We explore the welfare and policy implications of the model. We derive the marginal gains to a policy designer aiming to maximize (i) ex-ante adoption and (ii) welfare (i.e. the benevolent planner) across the network, via subsidizing the adoption of one agent. We show that the marginal impact of these targeted policies become independent of the particular target chosen from the target's coordination set. That is, optimal policy design becomes a problem of targeting a given coordination set rather than a particular agent.

For policies maximizing ex-ante adoption, the designer faces the following tradeoff. If she subsidizes adoption within large interconnected coordination sets where strategic contagion is relatively broad, the influence of the intervention on the targeted coordination set will be limited due to the stickiness of coordination. That is, while the intervention reaches a large set of agents, the impact of the intervention on each member decreases with the total number of agents coordination together (i.e. the size of the coordination set). We show that in the limit these effects perfectly balance, with the planner's objective reducing to a function of only the targeted coordination set's equilibrium cutoff.

The tradeoffs faced by the benevolent planner are more complex. The contagion-versus-sticky-coordination tradeoff remains. In addition, however, policy interventions also impose positive (expected) externalities on agents adjacent to the target's coordination set and who take lower cutoffs, though having no influence on their equilibrium behaviors. This establishes a fundamental wedge between the objectives of planners aiming to maximize adoption versus ex-ante welfare. Now, the benevolent planner must account for the direct impact on the target's welfare, the value from contagion across the target's coordination set, and the positive externalities to adjacent agents in other coordination sets. Despite this complexity, as with the adoption-maximizing target, the welfare-maximizing target is shown to depend only on which coordination set the target inhabits, and not the target's particular location within the coordination set. Strikingly, under homogenous values and some conditions on primitives, the adoption- and welfare-maximizing target coordination sets take opposite extremes, the former being the highest coordination set (where agents take the highest cutoff), and the latter being the lowest coordination set, which takes the lowest cutoff.

The paper is organized as follows. In the next section, we relate our paper to the relevant literature. Section 3 provides some examples to motivate our analysis. Section 4 introduces the model. Section 5 solves for the limiting equilibrium in cutoff strategies, and defines the notion of agent coordination sets. In Section 6, we characterize the equilibrium when the network structure provides all of the heterogeneity in the model. In Section 7, we address comparative statics with respect to intrinsic valuations, to formally describe sticky coordination and contained contagion. Section 8 discusses the welfare and policy implications for targeted adoption subsidies. Section 9 discusses extensions and applications to platform adoption, crime, and immigration policy. Finally, Section 10 concludes. All proofs can be found in the Appendix. An Online Appendix collects extensions and numerical solutions discussed in the sequel.

2 Related Literature

This paper adds to the growing literature on network games.[4] Ballester et al. (2006) [5], and more recently Bramoullé et al. (2014) [10] characterize conditions for equilibrium

[4]See Jackson (2008) [33] chapter 9, Jackson and Zenou (2015) [35] and Bramoullé and Kranton (2016) [9] for surveys.

existence and uniqueness when actions are continuous and best replies are linear.[5] Galeotti et al. (2010) [25] obtain multiplicity of equilibria in games under more general best replies, under incomplete and symmetric information of the extended network structure (beyond own degree).

The present paper takes strategic complements under incomplete information. While multiple equilibria again obtain under complete information, noisy information of a common fundamental state provides a unique equilibrium selection in the noiseless limit of our game. Also, in terms of policy implications, while the literature on network games advocates to target individual agents in terms of their centrality (*key players*), in the current paper, we show that one should target instead *key coordination sets*.

This paper also adds to a younger literature on network games with incomplete information. Calvó-Armengol et al. (2007) [11] and De Marti and Zenou (2015) [18] study the linear-quadratic setting of Ballester et al. (2006) [5] under the enrichment of a Bayesian game. Calvó-Armengol et al. (2015) [12], Leister (2017) [37] and Myatt and Wallace (2017) [45] incorporate endogenous investment in signal precision in these settings. And in a different vein, Hagenbach and Koessler (2010) [31] and Galeotti et al. (2013) [24] study cheap-talk in networks. Golub and Morris (2017a,b [29, 30]) study consistency and convergence in higher order expectations in Bayesian network games under linear best replies.

The current paper diverges from these contributions by focusing the analysis near and in the noiseless limit, and taking actions to be binary.

Carlsson and van Damme (1993) [13] first exhibited this selection devise for global games of two players and two actions.[6] Frankel et al. (2003) [22] extend the result to arbitrary games of strategic complements. In a two-sided environment, Morris and Shin (1998) [41] provide closed forms to their common limit-equilibrium cutoff, toward studying the interaction of a government defending a currency from a continuum of currency speculators.[7] Sákovics and Steiner (2012 [50]) study policy impact in a global game with a continuum of agents who value an agent-weighted average action. Dai and Yang (2017) [16] study a similar model to Sákovics and Steiner (2012) in which the continuum of agents carry private information regarding idiosyncratic costs of adoption, incorporating multiple cutoffs in the noiseless limit. In this setting, the authors focus on the role of organizations in mitigating miscoordination.

[5]These conditions involve bounding eigenvalues of transformations of the network's adjacency matrix.

[6]They show the risk-dominant equilibrium to be selected in these games.

[7]Related applications include crises and banks runs; see Dasgupta (2004) [17], Goldstein and Pauzner (2004, 2005) [27] [28], and Rochet and Vives (2004) [49].

Compared to this literature, our main contribution is to characterize properties of the cutoffs as a function of the network structure and then to partition players into coordination sets of players, i.e., set of players where all members take a common cutoff strategy and are path connected. We are also able to show under which condition of the network structure a single coordination set exists. Furthermore, our policy analysis contrasts adoption-based with welfare-based policies to establish a basic wedge between the two benchmarks, a wedge which only obtains under multiple cutoffs.[8]

Our results also bare on those of the network contagion literature. Chwe (2000) [14], Morris (2000) [40], and recently Jackson and Storms (2017) [32] study coordination games on a network under complete information[9], focusing on variants of network "cohesion" (Morris 2000) to characterize equilibrium adoption. These settings are similar to ours in the sense that agents decide to adopt or not under the presence of network externalities, but differ in the sense that, here, payoffs depend on a common unobserved state. While connectivity within agent sets similarly plays an important role in the present model, the global games selection of a unique equilibrium gives an elementary departure from these works.[10] Elliot et al. (2014) [21] and Acemoglu et al. (2015) [3] model the clearing of liabilities between institutions. The contagion of the ensuing model offers an alternative prediction to the spread of perturbations over the network, while incorporating strategic play, be it under a more elemental machinery.

3 Motivation

The basic question driving this paper is as follows. When agents' adoption of a technology affects the technology's value experienced by others, which agents will tend to adopt together? Consider for example the network of agents depicted in Figure 1. Assume that the value of the technology to any two connected agents, such as agents 1 and 2, increases when the agents simultaneously adopt. And, assume that all agents perfectly observe the state of the world, defined by a common component to the technology's value.

Now consider agents 1 and 3. Agent 3, like agent 2, takes a more central position

[8] Sákovics and Steiner find the optimal adoption-minimizing subsidy targets agents with high influence while being relatively uninfluenced by others.

[9] Chwe (2000) does incorporate two idiosyncratic states "willing" and "unwilling" (to adopt), but focuses on network properties sufficient for adoption by all willing agents independent of others' states.

[10] In a setting similar but more general than Morris (2000) [40] where an infinite population of players interact locally and repeatedly, Oyama and Takahashi (2015) [47] determine when a convention spreads contagiously from a finite subset of players to the entire population in some networks.

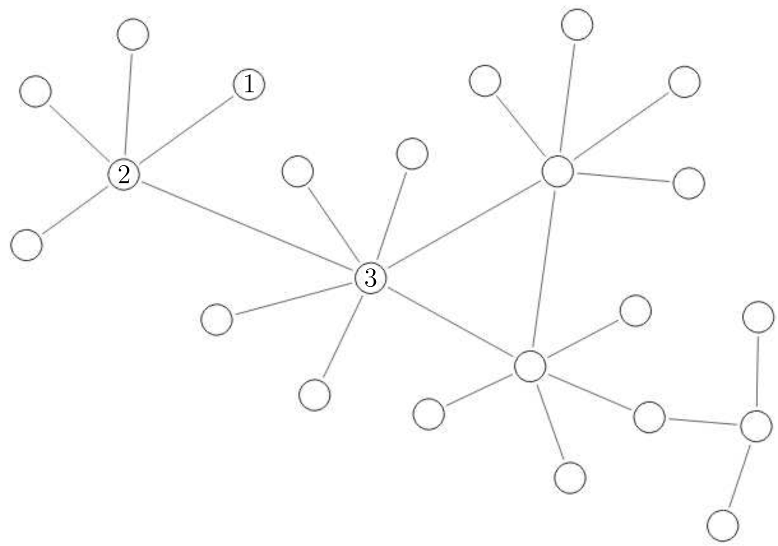

Figure 1: Who coordinates with whom?

in the network than does agent 1, who is positioned on the periphery. Given agent 3's advantageous position, having 7 times the degree of agent 1, will she adopt in strictly more states than agent 1? As is common in games with peer effects and two actions, multiple Nash equilibria obtain under complete information. And indeed, there will exist multiple equilibria in which player 3 and her neighbors (other than 2) adopt more often than agents 1 and 2, by coordinating together when agents 1 and 2 do not.[11]

As is well studied by the global games literature, if the quality of the technology (state of the world) is unknown and agents observe private signals of the state, with even the mildest noise, things change considerably. In particular, a unique equilibrium obtains upon introducing this mild strategic uncertainty into the environment. And in the noiseless limit, a unique Nash equilibrium of the complete information game is selected. What can we say about the properties of this equilibrium selection?

Currently, there is no prediction in the literature regarding exactly who, as a function of agents' position in the network, take a common action together in the noiseless limit. The main methodological contribution of this paper is to give an exact prediction by putting forward the notion of *coordination sets*, which gives an endogenous unique partitioning of the agents in a network. Agents within a coordination set are path connected within the coordination set and adopt the technology together.

The impact of incomplete information on synchronizing adoption in the network can

[11] In fact, there exists a continuum of such equilibria, each defined by the cutoff 1 and 2 coordinate on.

be extensive. As the sequel will show, all agents in tree networks (i.e. networks without any cycles) and in networks with at most one cycle (as in Figure 1) will adopt the technology simultaneously in the noiseless limit. As depicted in Figure 2, in the noiseless limit and for some unique, common threshold, all agents in the network will not adopt when the state is below this threshold, and will all simultaneously adopt upon the state rising above this threshold.

(a) low technology value: no adoption. (b) low technology value + "epsilon": all adopt.

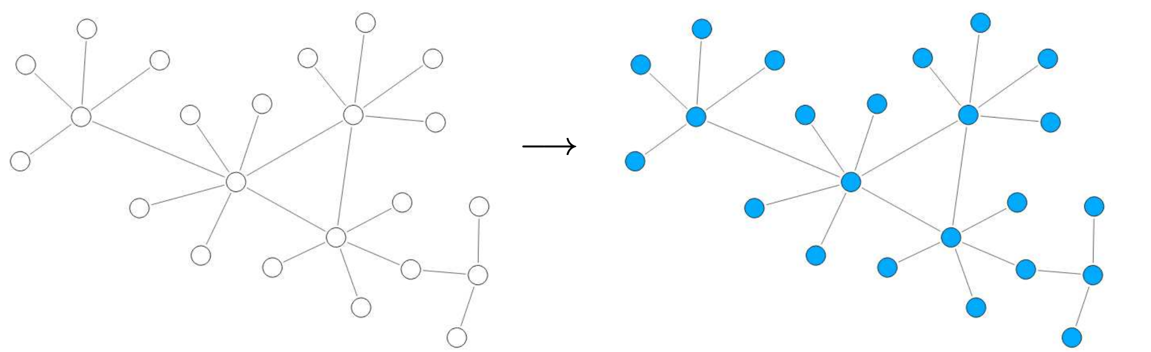

Figure 2: Global coordination.

The intuition behind this result is as follows. In equilibrium, when agents hold noisy signals of the technology's value, conditional expectations of others adopting must be (in some sense) consistent. This means that if one agent adopts under low states when a neighbor does not, the two neighbors must place probabilities zero and one, respectively, on the other adopting under these states. One can show that for the network structure in Figure 1 there is never a division of agents, with agents in one subset adopting and in the other not adopting, such that consistent expectations can be maintained. Therefore, in the noiseless limit, all agents must coordinate on adopting the technology together. In this paper, we will solve for the exact threshold at which this global adoption occurs.

Importantly, while this example may suggest that the network structure is inconsequential in determining who coordinates with whom, indeed the full structure of the network is crucial to deriving agents' coordination sets.[12] In particular, we show that multiple coordination sets obtain *if and only if* there is *imbalance* in average-local network effects across different locations of the network.[13] We show that once the coordination

[12]While the prior literature has focused on aggregative games to describe a common cutoff, here, the network of local interactions allows for a characterization of coordination sets.

[13]Section 6 formally defines our notion of network balance.

sets are determined from the network structure, this partition serves as a basic statistic shaping the properties of equilibrium play.

The implications for policy are reaching. First, if we empirically test coordination in networks, we need to have a unique equilibrium. Our framework supplies this. Second, we need to have predictions regarding who is and is not adopting under various states, and what properties of the environment affects this. Our framework supplies these relationships. The conclusions will be seen far distant from many policies suggested in the literature. In the context of microfinance, for example, Banerjee et al. (2013) advise adoption subsidies toward the agents with the highest eigenvector centrality. In Figure 1, agent 3 has a much higher eigenvector centrality than agent 1.[14] Contrary to these classical results, in Section 8, we show that optimal targeting is not about targeting individuals, but rather about targeting an optimal coordination set. If we follow eigenvector targeting, in our model, the policy impact in the example is exactly the same, independent of whether the subsidy goes to agent 3 (the center) or agent 1 (the periphery), or any other agent in this network. More generally, subsidies will increase the frequency of adoption by all agents within the target's coordination set, but at most increase the expected value of the technology enjoyed by agents (when they adopt) that are adjacent to the target's coordination set. Therefore, identifying the key coordination set requires accounting for large aggregate strategic effects within a coordination set while internalizing any external effects on adjacent agents' values. These basic tradeoffs become central to policy design. As a result, compared to the literature on games on networks (Jackson and Zenou (2015) [35]), which advocates to target *key players* in order to maximize adoption by others (Banerjee et al. (2013) [2]), here we put forward the importance of *key coordination sets*.

4 Model Setup

A finite set of agents N simultaneously choose whether or not to adopt a technology.[15] $a_i \in \{0, 1\}$ will denote agent i's choice to adopt. The components of the model are defined as follows. We reserve bold symbols to denote vectors (e.g., $\mathbf{a} := (a_1, \ldots, a_{|N|})$).

Payoffs. Payoffs from adopting the technology depend on the underlying fundamental

[14] Agent 3 has the largest eigenvector centrality 0.522, while agent 1 has eigenvector centrality 0.071.

[15] For the sake of the exposition, we use the example of technology adoption but, of course, all results subsume arbitrary binary action sets.

state θ, continuously distributed over bounded, interval support $\Theta \subseteq \mathbb{R}$. Moreover, the agents are connected via a network $\mathcal{G} = (N, E)$. E defines the set of edges between unordered pairs ij taken from N. We assume a connected and undirected graph: $i \in N_j$ if and only if $j \in N_i$, where $N_i := \{j : (i, j) \in E\}$ is the set of i's neighbors, and $d_i := |N_i|$ her degree. Then, each i obtains the following payoff from adopting ($a_i = 1$):

$$u_i(\mathbf{a}_{-i}|\theta) = v_i + \sigma(\theta) + \phi \sum_{j \in N_i} a_j \tag{1}$$

where $v_i \in \mathbb{R}$, $\sigma : \Theta \mapsto \mathbb{R}$, and $\phi > 0$. v_i gives the intrinsic (state independent) value to i from adopting, σ the state dependent value, with each of i's neighbors' adoption positively influencing the technology's value. $\sigma(\theta)$ is assumed to be differentiable and strictly increasing in θ. The network effect ϕa_j in (1) captures the positive externality that j's adoption imposes on i, while ϕ uniformly scales the size of network effects. The value to each agent from not adopting the technology is normalized to zero.

Dominance Regions. For each i, we assume v_i, σ and ϕ are such that there exist $\underline{\theta}_i$ and $\overline{\theta}_i$ such that $v_i + \sigma(\theta) + \phi d_i < 0$ when $\theta < \underline{\theta}_i$ and $v_i + \sigma(\theta) > 0$ when $\theta > \overline{\theta}_i$. Thus, there exist dominant regions $[\min \Theta, \underline{\theta}]$ and $[\overline{\theta}, \max \Theta]$, with $\underline{\theta} := \min_i\{\underline{\theta}_i\}$ and $\overline{\theta} := \max_i\{\overline{\theta}_i\}$, such that not adopting and adopting the technology (respectively) are dominant strategies for all players. When the realization of θ is common knowledge amongst agents, with σ continuous in θ and $\phi > 0$, there can exist a strictly positive measure of θ realizations within $[\underline{\theta}, \overline{\theta}]$ at which multiple pure strategy Nash equilibria occur.

Information Structure. In the perturbed game, θ is observed with noise by all agents. Each i realizes signal $s_i = \theta + \nu\epsilon_i$, $\nu > 0$, where ϵ_i is distributed via density function f and cumulative function F with support $[-1, 1]$. All signals are independently drawn across agents conditional on θ. For each $\nu > 0$, we write $G(\nu)$ the corresponding global game.[16,17]

[16]The assumption of a common noise structure is without loss of generality as the limit-equilibrium selection is robust to arbitrary, idiosyncratic F_i. Moreover, all results in the limit hold under unbounded supports (e.g. Gaussian state and noise).

[17]As is standard in the global game literature, we assume agents hold a dispersed prior of θ; see Sákovics and Steiner (2012 [50]) for discussion. This is inconsequential for the equilibrium selection and properties in the limit $\nu \to 0$.

5 Characterizing the Limit Equilibrium

$G(\nu)$ gives a Bayesian game of strategic complements between agents. Agent i chooses (possibly mixed) signal-contingent strategy $\pi_i : S \mapsto [0,1]$, mapping each signal realization to the likelihood i adopts. We write $\boldsymbol{\pi} := (\pi_1, \dots, \pi_N)$. For $\nu > 0$, define i's cutoff strategy at $c_i \in S$ by:

$$\pi_i(s_i) := \begin{cases} 1 & \text{if } s_i \geq c_i \\ 0 & \text{if } s_i < c_i \end{cases}.$$

Given cutoff strategy $\boldsymbol{\pi}_{-i}$ (at c_i) and conditional on signal realization s_i, i's expected payoff from adopting is given by:

$$\begin{aligned} U_i(\boldsymbol{\pi}_{-i}|s_i) &:= \mathbb{E}_\theta \left[\mathbb{E}_{\mathbf{s}_{-i}} \left[u_i(\mathbf{a}_{-i}|\theta) \,|\boldsymbol{\pi}_{-i}, \theta \right] \middle| \, s_i \right] \\ &= v_i + \mathbb{E}_\theta \left[\sigma(\theta) | s_i \right] + \phi \sum_{j \in N_i} \mathbb{E}_\theta \left[\mathbb{E}_{s_j} \left[\pi_j(s_j) | \, \theta \right] \middle| \, s_i \right] \end{aligned} \tag{2}$$

That is, i takes expectations of the technology's state-dependent value and of her neighbors' actions, conditioning on her private signal s_i. Then, a Bayesian Nash Equilibrium $\boldsymbol{\pi}^*$ of $G(\nu)$ in cutoff strategies satisfies $U_i(\boldsymbol{\pi}^*_{-i}|s_i = c_i^*) = 0$ for all $i \in N$, with each i indifferent between adopting and not adopting when observing signal s_i equal to her equilibrium cutoff c_i^*. Frankel et al. (2003) [22] Theorem 1 establishes uniqueness of a limiting equilibrium in general global games of strategic complements.[18] Indeed, in settings with binary actions, a pure Bayesian Nash Equilibrium in cutoff strategies obtains, which implies the unique limit equilibrium is also in cutoff strategies. Online Appendix B formally discusses existence and uniqueness of the limit equilibrium.

Any Bayesian Nash Equilibrium $\boldsymbol{\pi}^*$ can now be equivalently characterized by its cutoffs $\mathbf{c}^*$. Moreover, we can characterize the unique limit equilibrium $\lim_{\nu \to 0} \boldsymbol{\pi}^*$ of $G(0)$ by solving for the limiting state cutoffs $\boldsymbol{\theta}^* := (\lim_{\nu \to 0} c_i^*)_{i \in N}$, with each i choosing to adopt when θ rises above θ_i^*. Calculating cutoffs $\boldsymbol{\theta}^*$ requires finding a consistent set of limiting expectations, for each agent, on other agents' adoption choices. For this, denote $\mathbf{w}^*$ the limiting expectations placed on neighbors adopting in equilibrium $\boldsymbol{\pi}^*$ when each agent i realizes signal s_i equal to her equilibrium cutoff c_i^*. Precisely:

$$w_{ij}^* := \lim_{\nu \to 0} \mathbb{E}_{s_j} [\pi_j^*(s_j) | s_i = c_i^*] \in [0,1].$$

[18] In their generalized setting, this equilibrium may be in mixed strategies. Our contribution relative to this is to fully characterize a unique pure-strategy limit equilibrium in binary-action network games of strategic complements.

We give the following lemma.

Lemma 1. *For each $(i, j) \in E$, the following identity holds:*

$$w^*_{ij} + w^*_{ji} = 1. \tag{3}$$

Moreover, if $\theta^*_i < \theta^*_j$, then

$$w^*_{ij} = 0, \ \textit{and}\ w^*_{ji} = 1. \tag{4}$$

To interpret (3), consider the special case when $c_i = c_j = c^*$ near the noiseless limit. A signal realization of $s_i = c^*$ leaves i placing a fifty-fifty gamble on j adopting. This probability weighting persists in the noiseless limit, as captured by $w^*_i = w^*_j = 1/2$. The fact that these limiting likelihoods sum to one holds generally, including when $c^*_i \neq c^*_j$ in a neighborhood of the shared limit cutoff $\theta^*_i = \theta^*_j = \theta^*$. For (4), when $\theta^*_i < \theta^*_i$, then indeed, agent i places a zero percent likelihood on j adopting and j places a 100 percent likelihood on i *not* adopting when each realizes their respective cutoff.

Now, define the set of *feasible weighting functions* for $\mathcal{G}$:

$$\mathcal{W} = \{\mathbf{w} = (w_{ij}, (i, j) \in E) | w_{ij} \geq 0, w_{ji} \geq 0, w_{ij} + w_{ji} = 1; \forall (i, j) \in E\}.$$

Clearly, $\mathcal{W}$ is compact, convex, and isomorphic to $[0, 1]^{e(N)}$, where $e(N)$ denotes the number of links in $\mathcal{G}$. Note that (3) implies $\mathbf{w}^* \in \mathcal{W}$. For each $i \in N$, given intrinsic value v_i, scale factor ϕ and edges E we can define the affine mapping $\Phi_i : \mathcal{W} \to \mathbb{R}$:

$$\Phi_i(\mathbf{w}) =: v_i + \phi \sum_{j \in N_i} w_{ij}, \quad \forall i \in N. \tag{5}$$

Let $\Phi(\mathcal{W}) \subset \mathbb{R}^n$ denote the image of $\mathcal{W}$ under the mapping Φ. Given linearity of $\Phi(\cdot)$, $\Phi(\mathcal{W})$ is a compact, convex polyhedron. Denote $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$ and $||\mathbf{x} - \mathbf{y}|| := \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle}$ the Euclidean norm. The following theorem is used to calculate the limit cutoffs $\boldsymbol{\theta}^*$.

Theorem 1. *For any $\mathbf{v}$, ϕ, and network $\mathcal{G}$, the equilibrium limit cutoffs $\boldsymbol{\theta}^*$ are given by:*

$$\sigma(\theta^*_i) + q^*_i = 0, \quad \forall i \in N, \tag{6}$$

where $\mathbf{q}^* = (q^*_1, \cdots, q^*_n)$ is the unique solution to:

$$\mathbf{q}^* = \operatorname*{argmin}_{\mathbf{z} \in \Phi(\mathcal{W})} ||\mathbf{z}||. \tag{7}$$

Theorem 1 gives the equilibrium cutoff value for each agent i. It is very powerful because it determines exactly which agents will adopt and which ones will not for each θ, for any arbitrary network structure $\mathcal{G}$ and any vector of intrinsic valuations $\mathbf{v}$. The solution of Theorem 1 is expressed in terms of $\mathbf{q}^*$, which maps one-to-one to and is monotonically decreasing with $\boldsymbol{\theta}^*$, as defined by (6). Strikingly, $\mathbf{q}^*$ solves a simple quadratic program with linear constraints, as defined by (7). $\mathbf{q}^*$ maps back to weighting matrix $\mathbf{w}^*$, via $\Phi(\cdot)$. That is, $\mathbf{q}^* = \Phi(\mathbf{w}^*)$. Note that Lemma 1 guarantees that $\mathbf{q}^*$ satisfies the necessary condition of belief consistency. Even more, these conditions with the agents' indifference conditions are sufficient for $\mathbf{q}^*$ to solve program (7) of Theorem 1.

More generally, to characterize the (noiseless) limit equilibrium cutoffs, we proceed in three steps. First, we have Lemma 1, which characterizes a key property of beliefs for any two players i, j. Then, we use this belief property to pin down the equilibrium signal cutoffs in the limit (as $\nu \to 0$). Such cutoffs need to satisfy the system of indifference conditions:

$$v_i + \sigma(\theta_i^*) + \phi \sum_{j \in N_i} \lim_{\nu \to 0} \mathbb{E}[\pi_j^*(s_j) | s_i = c_i^*] = 0, \quad \forall i \in N.$$

The third step is stated in Theorem 1. To characterize equilibrium cutoffs, we use the above property of beliefs (Lemma 1) to show that solving the system of indifference conditions is equivalent to solving a quadratic program with linear constraints. Indeed, the minimization program (7) in Theorem 1 solves for a set of cutoffs by way of *minimizing* miscoordination in adoption across agents, in a manner which is compatible with a set of consistent beliefs.

For any agent subset $S \subseteq N$, denote E_S the subset of edges in E corresponding with the subgraph $\mathcal{G}_S := (S, E_S)$ of $\mathcal{G}$ restricted to vertices S.[19] The limit equilibrium $\lim_{\nu \to 0} \boldsymbol{\pi}^*$ must then define an ordered partition $\mathcal{C}^* := (C_1^*, \ldots C_{\bar{m}^*}^*)$ of N (i.e. $\cup_m C_m^* = N$ and $C_m^* \cap C_{m'}^* = \emptyset$ for $m \neq m'$). The following notion of a *coordination set* will account for both coordination and miss-coordination in general network structures.

Definition 1 (Coordination sets)**.** *The limit equilibrium $\vec{\boldsymbol{\pi}}$ maps to a unique ordered partition $\mathcal{C}^* := (C_1^*, \ldots C_{\bar{m}^*}^*)$ of N satisfying:*

1. *common adoption: for each m, $C_m^* \mapsto \theta_m^* \in \Theta$ with $\theta_i^* = \theta_j^* = \theta_m^*$ for each $i, j \in C_m^*$, and $\theta_m^* \leq \theta_{m'}^*$ for each $m < m'$,*
2. *within-set path connectedness: for each m, $\mathcal{G}_{C_m^*}$ is connected,*
3. *coarse partitioning: for each $m \neq m'$ such that $\theta_m^* = \theta_{m'}^*$, $E_{C_m^* \cup C_{m'}^*} = E_{C_m^*} \cup E_{C_{m'}^*}$.*

[19]Precisely, $(i, j) \in E_S$ if and only if $i, j \in S$ and $(i, j) \in E$.

Each C_m^* defines a *coordination set* of agents. By condition 1, each agent within a coordination set shares the same cutoff, which we are free to order when defining $\mathcal{C}^*$. By condition 2, these agents are connected via paths within their coordination set. Condition 3, which establishes coordination sets sharing the same cutoff as being disconnected, ensures that each set is maximally inclusive without violating Conditions 1 and 2.[20]

The agents' coordination sets constitute the basic architecture and accompanying properties of the limit equilibrium. Importantly, the grouping of agents according to Definition 1 is without loss of generality, as the exhaustive partition (i.e. $\mathcal{C} = \{\{i\}; i \in N\}$) satisfies conditions 1, 2 and 3, with agents coordinating away from each other by taking distinct cutoffs. When convenient, $m(i)$ will denote i's coordination set: $i \in C^*_{m(i)}$.

Upon determining the limiting coordination sets $\mathcal{C}^*$ using Theorem 1 and Definition 1, we can easily recover $\mathbf{q}^*$, as follows. Through the sequel, $d_i(S) := |N_i \cap S|$ will denote the *within-degree* of i, or the number of edges between i and members of agent set S. Define for any disjoint agent sets S and S':

$$e(S, S') = \sum_{i \in S} d_i(S'),$$

the number of edges from S to S'. And for any agent set S, define:

$$e(S) = \frac{1}{2} \sum_{i \in S} d_i(S),$$

the number of edges between members of S. $v(S) := \sum_{i \in S} v_i$ denotes the sum of intrinsic values among members of S. For each $C_m^* \in \mathcal{C}^*$ denote $\underline{C}_m^* := \cup_{m' < m} C_{m'}^*$, which includes all neighbors to C_m^* taking cutoffs below θ_m^* . We can now give a reduced characterization of $\mathbf{q}^*$.

Proposition 1. *For each $C_m^* \in \mathcal{C}^*$, each $q_i^* = q_m^*$, $i \in C_m^*$, where:*

$$q_m^* = \frac{v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))}{|C_m^*|}. \tag{8}$$

That is, each q_m^* gives exactly the average (across $i \in C_m^*$) of v_i, plus ϕ times the average number of links to agents taking strictly lower cutoffs, plus ϕ times one-half the average within-degree $d_i(C_m^*)$. With $\theta_i^* = \sigma^{-1}(-q_i^*)$ for each $i \in N$ (i.e. condition (6)), this

[20] Conditions 1 through 3 pin only a partial ordering of $\{C_1, \ldots, C_{\bar{m}^*}\}$, and thus there can be multiple orderings satisfying the conditions. It is without loss to select one arbitrarily.

provides a calculation of $\boldsymbol{\theta}^*$ solely in terms of counting intrinsic values and degrees, both within and across coordination set. Strikingly, Proposition 1 shows that, while $\mathcal{G}$ plays a key rule in determining the limit partition $\mathcal{C}^*$, upon conditioning on $\mathcal{C}^*$ the network structure within coordination sets plays no role in determining limiting cutoffs. Precisely, given $v(C_m^*)$, $e(C_m^*, \underline{C}_m^*)$, and $e(C_m^*)$, moving the position of links within C_m^* carries no impact on θ_m^* as long as the coordination sets are not affected. In other words, while the structure of $\mathcal{G}$ plays a global role determining who coordinates with whom, its role is muted at the local level.

Let us summarize the main results of this section, which we believe is one of the main contributions of this paper. First, in Lemma 1, we characterize a key property of beliefs for any two players i, j, that is: $\mathbb{E}[\pi_i^*(s_i)|s_j = c_j^*] + \mathbb{E}[\pi_j^*(s_j)|s_i = c_i^*] = 1$, in the limit, with $\lim_{\nu\to 0} \mathbb{E}[\pi_i^*(s_i)|s_j = c_j^*] = 0$ if $\theta_i^* > \theta_j^*$. Then, we use this belief property to pin down the equilibrium signal cutoffs in the limit; such cutoffs need to satisfy a system of indifference conditions. Theorem 1 characterizes these equilibrium cutoffs, showing that consistency in beliefs with the system of indifference conditions imply that the cutoffs solve a simple quadratic minimization program with linear constraints. This program effectively minimizing miscoordination over the network in a manner which respects belief consistency.

It turns out that the vector of cutoffs, $\mathbf{q}^*$, is a projection of the origin onto the compact, convex space $\Phi(\mathcal{W})$, which is the image of $\mathcal{W}$ under the mapping Φ: $\mathbf{q}^* = \mathbf{Proj}_{\Phi(\mathcal{W})}[\mathbf{0}]$, for $\mathbf{0}$ the vector of zeros in $\mathbb{R}^n$. In the Online Appendix C.1, we provide a simple example for the dyad network. In this example, solving the minimization program in (7) in Theorem 1 is equivalent to minimize the distance between $\Phi(\mathcal{W})$ and the 45 degrees line, that is where $q_1 = q_2$. By setting $q_1^* = q_2^*$ when possible, the two agents can capitalize on each others' network effects in all states in which they adopt.

In the Online Appendix C.2, we provide an algorithmic approach for calculating the limiting coordination sets (see Algorithm 1), by recursively constructing the sets beginning with C_1^*. Proposition C1 formally establishes the relationship between Algorithm 1 and Theorem 1, showing that the two approaches can be viewed as dual problems. Theorem 1 calculates $\mathbf{q}^*$ yielding the partition $\mathcal{C}^*$ as a bi-product, while Algorithm 1 constructs $\mathcal{C}^*$ yielding $\mathbf{q}^*$ as a bi-product.

Finally, in Proposition 1, we calculate the cutoff values as a function of the coordination sets. Indeed, for any set S of connected agents that converge on a common cutoff θ^*, we can average over expected network effects and apply the belief property to obtain a limiting average network externality between members of S when θ^* is observed:

$$\frac{\sum_{i\in S}\sum_{j\in N_i\cap S}\phi\mathbb{E}[\pi_j^*(s_j)|s_i=c_i^*]}{|S|}\xrightarrow{\nu\to 0}\phi\frac{\#\ edges\ between\ agents\ in\ S}{2|S|} \tag{9}$$

If we take $S=C_m^*$, and sum up the indifference condition $q_i=v_i+\phi\sum_{j\in N_i}w_{ij}$, we obtain that:

$$|C_m|q_m^*=\sum_{i\in C_m^*}q_i=\sum_{i\in C_m^*}v_i+\phi\sum_{i\in C_m^*}\sum_{j\in N_i\cap C_m^*}w_{ij}+\phi\sum_{i\in C_m^*}\sum_{j\in N_i\cap(N\setminus C_m^*)}w_{ij}$$

The first term is just $v(C_m^*)$ in (8) while the second term is $\phi e(C_m^*)$ by (9). For the third term, we need to differentiate between neighbors of agent i who take higher versus lower cutoffs. By Lemma 1, neighbors with strictly lower cutoff contribute exactly 1 to the sum while neighbors with strictly higher cutoff contribute exactly 0 to the sum. As a result, the third term gives the total number of edges from C_m^* to $\underline{C}_m^*$. Therefore, as stated in Proposition 1, we can retrieve limit cutoffs only using the information about the number of links between coordination sets.

6 Network Characterizations

Throughout this section, we assume *homogenous intrinsic values*, that is, $v_i=v$ for each i. By imposing such homogeneity, the structure of $\mathcal{G}$ solely determines the limiting coordination sets amongst agents.

6.1 Determining coordination sets

Let us provide necessary and sufficient conditions for a single coordination set in the network.

Proposition 2 (Single coordination set). *Under homogeneous intrinsic values, a single coordination set exists (i.e. $\mathcal{C}^*=\{C_1\}$) if and only if the network is* ***balanced****, in the sense that for every nonempty $S\subset N$,*

$$\frac{e(S)}{|S|}\le\frac{e(N)}{|N|}. \tag{10}$$

Condition (10) says that a network $\mathcal{G}$ is *balanced* if the average degree of each subnetwork

$\mathcal{G}_S$ is no greater than the average degree of the original network $\mathcal{G}$. When $\mathcal{G}$ is balanced, we see from (8) that the common cutoff in the network is $\theta_1^* = \sigma^{-1}(-v - \phi\frac{e(N)}{|N|})$.[21]

We can apply Proposition 2 to show a unique coordination set for the following families of network structures. We say network $\mathcal{G}$ is *regular* if $d_i = d$ for all i. A *tree* is any connected network without cycles. We say network $\mathcal{G}$ is a *regular-bipartite network* with disjoint within-set symmetric agent sets B_1 and B_2, with $B_1 \cup B_2 = N$ and of sizes $n_s := |B_s|$ and degrees $d_s := d_i,\ i \in B_s$, for sides $s = 1, 2$. Note that regular-bipartite networks satisfy $e(N) = n_1 d_1 = n_2 d_2$.

Proposition 3 (Single coordination set: examples)**.** *Under homogenous intrinsic values, there exists a single coordination set if $\mathcal{G}$ takes at least one of the following properties:*

1. *is a regular network, or*
2. *is a tree network, or*
3. *is a regular-bipartite network, or*
4. *has a unique cycle, or*
5. *has at most four agents.*

Proposition 3 exhibits the striking extent to which network-wide coordination may obtain. Members of all trees, regardless of their size and complexity, adopt using a common limit cutoff. Parts 2 and 4 establish the existence of at least two distinct cycles in $\mathcal{G}$ as a necessary condition for multiple limit cutoffs to obtain in equilibrium. This establishes trees as the family of network structures exhibiting the highest limit cutoffs. Still, regular-bipartite networks (and regular networks) may carry arbitrary numbers of cycles, yet all of these structures yield a unique coordination set.[22]

Where peer effects within subsets of agent are sufficiently imbalanced across the network, multiple coordination sets arise. The next proposition establishes two basic properties of the partition $\mathcal{C}^*$ under homogeneous intrinsic values. Denote $\hat{\mathbf{q}}^*$ to give the $\mathbf{q}^*$ at $\mathbf{v} = \mathbf{0}$ and $\phi = 1$.

Proposition 4 (Limit partition homogeneity)**.** *Under homogeneous intrinsic values, $\mathcal{C}^*$ is independent of v and of ϕ. Moreover, $\mathbf{q}^* = v\mathbf{1} + \phi\hat{\mathbf{q}}^*$.*

[21] See Online Appendix D.3 for the analogous condition incorporating heterogeneous intrinsic values.

[22] Proposition 3 part 5 can be extended to show that $|C_1^*| \geq 4$ whenever $|N| \geq 4$; see Proposition D2 of Online Appendix D.1. Remark 2 of that section provides general bounds on θ_1^* for tree and regular-bipartite networks.

Scaling the size of valuations or network effects has no effects on the limit partition. Moreover, $\mathbf{q}^*$ is linearly augmented by the size of values v and of network effects ϕ. Again, we see that $\mathcal{C}^*$ constitutes the basic structure of the limit equilibrium, in that coordination across players is invariant to the relative strengths of intrinsic values, state-dependent value and peer effects.

To summarize, in order to determine the (possible) different coordination sets in a network, one needs to check that the balanced condition given in (10) is verified.[23] If this condition is verified for each subset of agents, then we know that there is a unique coordination set with a common cutoff given by (8). If this condition is not satisfied for at least a subset of agents, then one can either use Theorem 1, which gives an exact cutoff value for each agent, leading to the determination of agents' coordination sets, or, use the Algorithm 1 given in the Online Appendix C.2 to determine the coordination sets. The following examples illustrate the first approach, and show how multiple coordination sets can arise. We focus on star and simple core-periphery networks.

Example 1. *Figure 3 gives the star and three core-periphery networks of differing core sizes. In each case we apply Proposition 2, focusing on agent sets which are symmetric over their respective agents. For these cases, the relevant* $e(S, S')$ *reduce to* $d_i(S')$ *and* $e(S)$ *to* $d_i(S)$ *for each* $i \in S$. *We set* $v = 0$ *and* $\phi = 1$, *giving* $\mathbf{q}^* = \hat{\mathbf{q}}^*$.[24]

For the star, if multiple coordination sets were to exist, the most natural case is for the center to take a strictly lower cutoff to the periphery. Defining agent set $S = \{c\}$ *we see that* (10) *is satisfied with:*

$$\frac{e(\{c\})}{|\{c\}|} = d_c(\emptyset) = 0 < \frac{3}{4} = \frac{e(N)}{|N|}. \tag{11}$$

This implies that when the center takes a strictly lower cutoff in the limit, it does not enjoy strictly strictly greater expected network effects than the periphery: a contradiction under homogenous intrinsic values. Upon establishing (10) *for all* S *(inequalities which are only more easily satisfied), we establish that all members of the star indeed coordinate together. Note that the analogous inequalities to* (11) *hold for any arbitrary number of peripheral agents, establishing that all agents of star networks coordinate on a common cutoff in the noiseless limit (Proposition 3).*

For the triad-core-periphery network depicted, set $S = \{1c, 2c, 3c\}$. (10) *is now weakly*

[23] This is when agents are ex ante identical in terms of intrinsic valuation of the technology; when they are ex ante heterogeneous, condition (D2) in the Online Appendix D.3 provides the equivalent condition.

[24] Given independence of $\mathcal{C}^*$ in $v\mathbf{1}$ and ϕ by Proposition 4, this is without significant loss of generality.

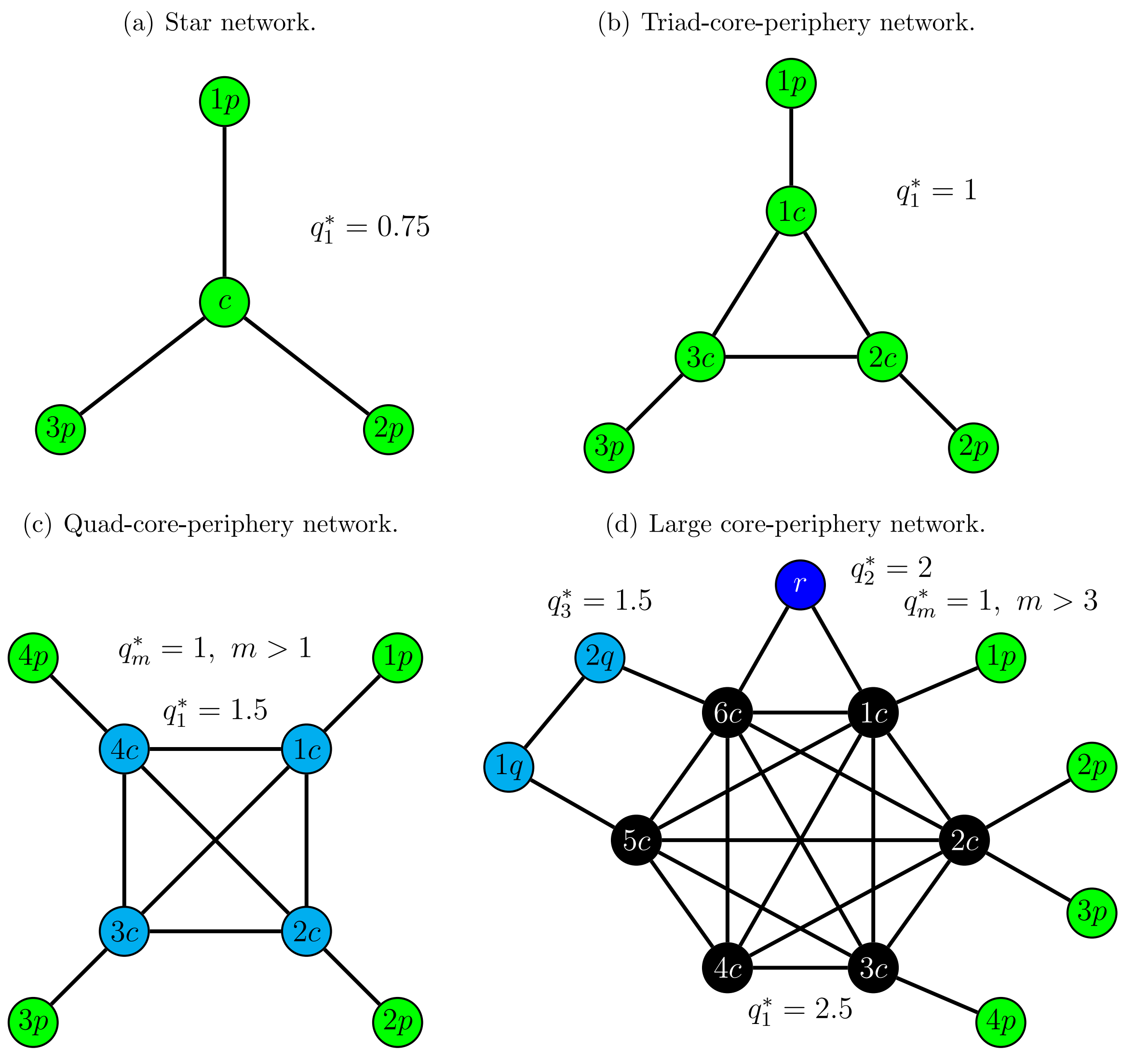

Figure 3: Coordination and network structure.

satisfied:

$$\frac{e(\{1c, 2c, 3c\})}{|\{1c, 2c, 3c\}|} = \frac{3}{3} = \frac{6}{6} = \frac{e(N)}{|N|}.$$

That is, the core and periphery hold equivalent expected network effects when the core takes a lower cutoff. As with the star, this violates the premise that the core takes a lower cutoff.

Once the size of the core exceeds three, as with the quad-core-periphery network, the expected network effects within the core suffice for it to break away from the periphery.

We violate (10) *by setting* $S = C_1^* = \{1c, \ldots, 4c\}$*:*

$$\frac{e(C_1^*)}{|C_1^*|} = \frac{6}{4} > \frac{10}{8} = \frac{e(N)}{|N|},$$

for each $j = 1, \ldots, 4$, *and thus each periphery agent is in its own coordination set,* C_2^* *through* C_5^**.*

This is similar for the Large core-periphery network. Here, we can use Theorem 1,[25] *which calculates the* q_m^**'s provided in Figure 3, which from Definition 1 imply that* $C_1^* = \{1c, \ldots, 6c\}$, $C_2^* = \{r\}$, $C_3^* = \{1q, 2q\}$, *with each periphery agent* jp, $j = 1, \ldots, 4$ *inhabiting their own coordination sets* $C_4^*, \ldots, C_7^*$*. Note that the number of peripheral cliques connecting to the core, each taking a local structure depicted in Figure 3, is inconsequential to the equilibrium cutoff of the core. This is precisely because core agents, upon observing their cutoff, place probability zero on all periphery agents adopting.*

Extending beyond Example 1, consider any core-periphery structure with regular core of degree d_c and size n_c, and with n_p periphery agents, each connected to k core agents symmetrically. This graph is balanced if and only if $d_c \leq 2k$. Either the core is not very connected, or the number of links to the core is very large. Otherwise the periphery agent will have a strictly higher cutoff (i.e. adopt less often) than the core agents.

6.2 Coordination in real-world networks

Here we explore our model's prediction in four examples of small real-world networks. We consider three components from the "help decision" network[26] in rural India studied in Banerjee et al. (2013) [2], and the friendship network of adolescents in the United States sourced from the Add Health data set. Figure 4 depicts coordination in each of these networks. In each network, v_i is set to zero for each i and $\phi = 1$. Each coordination set's q_m^* (from Theorem 1) is provided with each figure and different colors indicate different coordination sets.

We see that multiple coordination sets can obtain in small networks. For example, with the Banerjee et al. (2013) data, in network 2, there are six coordination sets from only twenty agents in total. Note the presence of one cycle in the Add Health network, which implies that agents coordinate on a common cutoff (see Proposition 3). Applying

[25] Algorithm 1 given in the Online Appendix C.2 can alternatively be used; the Online Appendix steps through the algorithm for the Large core-periphery network example.

[26] The exact question is: "If you had to make a difficult personal decision, whom would you ask for advice?".

(10) of Proposition 2, in this network, an equilibrium $q_1^* = 1$ obtains given that the number of links is equal to the number of agents. Upon deleting one link within the cycle, global coordination would persist with q_1^* dropping to $\frac{e(N)}{|N|} = \frac{22}{23}$.

These real-world network examples show that it is not immediately apparent which agents will coordinate by a casual inspection of the network's structure. For example, the Add Health network is as irregular as the other networks but the adoption decisions are very different. Our analysis demonstrates that network effects are crucial in adoption decisions, which are captured by coordination sets. The latter, however, are not derived from standard centrality measures, network density, clustering coefficient, etc., used in the network literature. Indeed, in standard network games with strategic complementarities (e.g. Ballester et al. (2006) [5]), actions are aligned with individual's centrality. Our analysis changes this perspective by looking at a more aggregate measure: the coordination set. For adoption decisions, it is not the individual centrality that matters but the coordination set the agent belongs to. Even more, as the following section shows, the coordination set a player belongs to will define exactly how she responds to policies targeting individuals' incentives to adopt.

7 Intrinsic Valuations Comparative Statics

In what follows, we reintroduce heterogeneous v_i. Our balanced-network characterization, Proposition 2, generalizes to this more general framework; see Proposition D4 of Online Appendix D.3.

We consider changes to intrinsic values $\mathbf{v}$, which may be the result of exogenous factors or the introduction of adoption-subsidization policies. In particular, we would like to investigate how changes in intrinsic values to one agent reverberate through that agent's entire coordination set. The first result shows that each member adjusts their cutoffs in step, independent of their network position within the coordination set.

Proposition 5 (Local contagion)**.** *In the limit, the mapping $\mathbf{q}^*(\mathbf{v})$ is piecewise linear, Lipschitz continuous, and monotone. For generic $\mathbf{v}$, when $i, j \in C_m$ and $k \notin C_m$, then:*

$$\frac{\partial q_j^*}{\partial v_i} = \frac{1}{|C_m|}, \qquad \textit{and} \qquad \frac{\partial q_k^*}{\partial v_i} = 0, \tag{12}$$

This proposition shows that increasing v_i, the instrinsic value for the technology of agent i belonging to coordination set C_m, reduces the common cutoff value θ_m^* for all agents

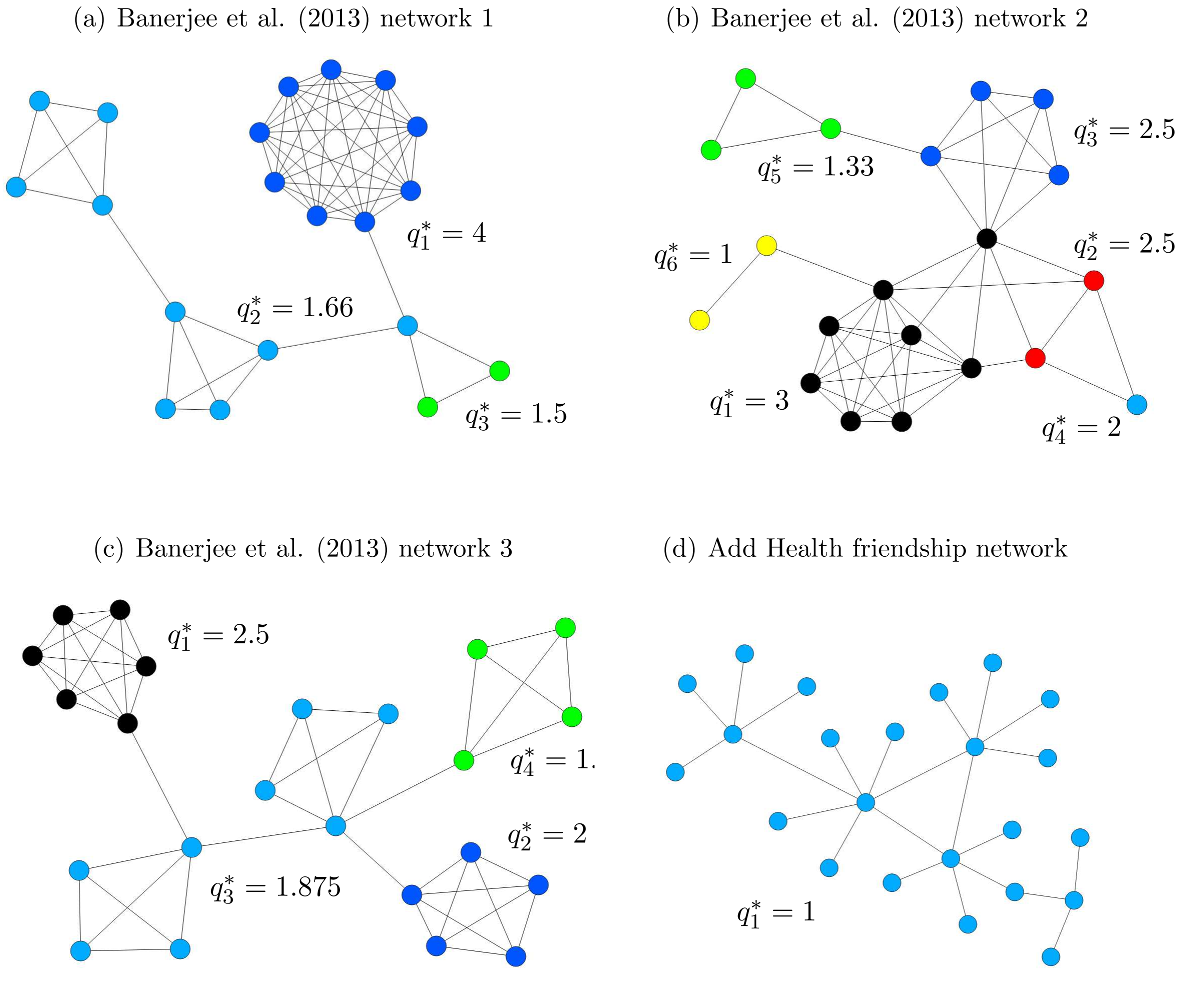

Figure 4: Coordination in real-world networks.

in the coordination set C_m so that all these individuals are now more likely to adopt.[27] Strikingly, the effect of increasing v_i lowers cutoffs at a rate inversely proportional to the size of the coordination set, $|C_m|$. That is, as the number of agents coordinating together increases, with each additional agent imposing strategic uncertainty on other members near the limit, the coordination set responds more slowly to a given increase in v_i. Another observation is that, for any $i, i' \in C_m$, we have $\frac{\partial q_l^*}{\partial v_i} = \frac{\partial q_l^*}{\partial v_{i'}} = \frac{1}{|C_m|}$ if $l \in C_m$, $\frac{\partial q_l^*}{\partial v_i} = \frac{\partial q_l^*}{\partial v_{i'}} = 0$ if $l \notin C_m$. In other words, the comparative statics results for any two agents in the coordination set C_m are exactly the same (see Proposition 7 for an illustration of this result to some policy problems). Again we see the local network

[27]Note the one-to-one inverse relationship between q_j^* and θ_j^*, given by $\sigma(\theta_j^*) + q_j^* = 0$, so that when partial $\frac{\partial q_j^*}{\partial v_i}$ exists, $\frac{\partial \theta_m^*}{\partial v_i} = \frac{\partial q_m^*}{\partial v_i} \frac{-1}{\sigma'(\theta_j^*)}$.

structure becomes muted in the limit, with (12) dependent on network $\mathcal{G}$ only through it's determination of limit partition $\mathcal{C}^*$.

Proposition 5 further establishes that a local change in the intrinsic value of the technology to agent i carries only an influence on members of i's coordination set, while having zero influence on adoption strategies of other coordination sets. The intuition is straight forward: for each $C^*_{m'}$ with $\theta^*_{m'} \neq \theta^*_m$, when members of $C^*_{m'}$ observe signals $s_{i'} \approx s^*_{i'}$ in the perturbed game $G(\nu)$, members of C^*_m are either all adopting or all not adopting the technology, depending on $m < m'$ or $m > m'$, respectively. Thus, while v_i carries influence on cutoffs within C^*_m, these strategic effects carry zero repercussion for coordination within $C^*_{m'}$.

The fact that $\frac{\partial q^*_j}{\partial v_i} > 0$ from (12) implies that $\frac{\partial c^*_j}{\partial v_i} < 0$ near the limit, by equilibrium cutoffs $\mathbf{c}^*$ continuously differentiable in $\mathbf{v}$ and in ν. Moreover, we can show that the discontinuous drop-to-zero in contagion across coordination sets persists near the limit.[28]

Remark 1. *Near the limit, for $k \notin C_{m(i)}$, $\frac{\partial c^*_k}{\partial v_i} = 0$ when $\nu < \bar{\nu}$ for some $\bar{\nu} > 0$.*

Regardless of the proximity of any two (connected) coordination sets' limit cutoffs, as ν diminishes the noise supports of agents positioned across the two coordination sets must ultimately separate, leaving each agent's cutoff locally invariant to subsidies to a member of the other set.[29]

The limiting coordination sets inevitably adjust as $\mathbf{v}$ is significantly shifted in a given direction. Lipschitz continuity in $\mathbf{q}^*(\mathbf{v})$ assures that the limit cutoffs do not discontinuously jump as we continuously change $\mathbf{v}$, including when the coordination sets adjust. The next result shows that as the network interactions are strengthened, the range of intrinsic values that support coordination amongst agents expands. This characterizes a *stickiness* in coordination as a result of network effects.

Take any $\mathcal{G}$, $\mathbf{v}$ and ϕ, and resulting limit partition $\mathcal{C}^*$. Then, for any $i \in N$ denote:

$$\begin{aligned} \hat{v}^*_i(\mathbf{v}_{-i}) &:= \text{argmax}\{v_i : \theta^*_i = \theta^*_j, j \in C^*_{m(i)} \backslash \{i\}; \mathbf{v}_{-i}\}, \\ \underset{\smile}{v}^*_i(\mathbf{v}_{-i}) &:= \text{argmin}\{v_i : \theta^*_i = \theta^*_j, j \in C^*_{m(i)} \backslash \{i\}; \mathbf{v}_{-i}\}. \end{aligned}$$

That is, $[\underset{\smile}{v}^*_i(\mathbf{v}_{-i}), \hat{v}^*_i(\mathbf{v}_{-i})]$ gives the ranges to i's intrinsic values that support i and members of $C^*_{m(i)} \backslash \{i\}$ (for at least one $j \in C^*_{m(i)} \backslash \{i\}$) coordinating on the same limiting

[28] Remark 1 requires agents' signal noise to be bounded. If, instead, signal noise follows a Gaussian distribution, for example, Proposition 5 obtains only in the limit.

[29] A formal proof of Remark 1 is provided with Online Appendix D.3.

adoption cutoff, holding $\mathbf{v}_{-i}$ fixed.[30] When $C^*_{m(i)} \backslash \{i\}$ coordinate on a common cutoff $\theta^*_{m(i)}$ over all $v_i \in (\hat{v}^*_i(\mathbf{v}_{-i}), \underset{\smile}{v}^*_i(\mathbf{v}_{-i}))$, then $\hat{v}^*_i(\mathbf{v}_{-i}) = \text{argmax}\{v_i : \theta^*_i = \theta^*_{m(i)}; \mathbf{v}_{-i}\}$ and $\underset{\smile}{v}^*_i(\mathbf{v}_{-i}) := \text{argmin}\{v_i : \theta^*_i = \theta^*_{m(i)}; \mathbf{v}_{-i}\}$.[31] The next result shows that $\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i})$ increases with the number of links between i's and her coordination set.

Proposition 6 (Sticky coordination)**.** *Take coordination set $C^*_m \in \mathcal{C}^*$ with $|C^*_m| > 1$. Then for each $i \in C^*_m$:*

$$\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i}) \geq \phi d_i(C^*_m). \tag{13}$$

When $\mathcal{C}^*$ is constant for $v_i \in (\hat{v}^*_i(\mathbf{v}_{-i}), \underset{\smile}{v}^*_i(\mathbf{v}_{-i}))$, then:

$$\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i}) = \frac{|C^*_m|}{|C^*_m| - 1} \phi d_i(C^*_m). \tag{14}$$

Expression (13) establishes that $\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i})$ is strictly positive and bounded below by ϕ times the number of neighbors i has in C^*_m. i's coordination with agents in $C^*_{m(i)}$ becomes more robust as network effects increase, either through an increase in ϕ or from additional links placed between i and members of $C^*_{m(i)}$. When $\mathcal{C}^*$ is constant over $v_i \in (\hat{v}^*_i(\mathbf{v}_{-i}), \underset{\smile}{v}^*_i(\mathbf{v}_{-i}))$ (that is, all coordination sets do not change as v_i varies in the interval $[\underset{\smile}{v}^*_i(\mathbf{v}_{-i}), \hat{v}^*_i(\mathbf{v}_{-i})]$), then $\hat{v}^*_i - \underset{\smile}{v}^*_i$ scales linearly with $d_i(C^*_{m(i)})$, with slope increasing in ϕ and the size of $C^*_{m(i)}$.[32] This proposition thus shows that, when social interactions in the network increase –either in number or in scale– the ranges of intrinsic values that support coordination amongst agents expand.

To interpret (13) and $\phi d_i(C^*_{m(i)})$ as an underlining lower bound to $\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i})$, consider the analogues to $\hat{v}^*_i(\mathbf{v}_{-i})$ and $\underset{\smile}{v}^*_i(\mathbf{v}_{-i})$ near the limit:

$$\begin{aligned} \hat{v}^*_i(\mathbf{v}_{-i}, \nu) &:= \text{argmax}\{v_i : |\theta^*_i - \theta^*_j| < 2\nu, j \in C^*_{m(i)} \backslash \{i\}; \mathbf{v}_{-i}\}, \\ \underset{\smile}{v}^*_i(\mathbf{v}_{-i}, \nu) &:= \text{argmin}\{v_i : |\theta^*_i - \theta^*_j| < 2\nu, j \in C^*_{m(i)} \backslash \{i\}; \mathbf{v}_{-i}\}, \end{aligned}$$

which obtain $\lim_{\nu \to 0} \hat{v}^*_i(\mathbf{v}_{-i}, \nu) = \hat{v}^*_i(\mathbf{v}_{-i})$ and $\lim_{\nu \to 0} \underset{\smile}{v}^*_i(\mathbf{v}_{-i}, \nu) = \underset{\smile}{v}^*_i(\mathbf{v}_{-i})$. When $v_i = \hat{v}^*_i(\mathbf{v}_{-i}, \nu)$ in the perturbed game $G(\nu)$, $c^*_i < c^*_j$ for each $j \in N_i \cap C^*_{m(i)}$, and thus the likelihoods that i and j place on the other adopting –when realizing signals equal to their respective equilibrium cutoffs– equal zero and one, respectively. When $v_i = \underset{\smile}{v}^*_i(\mathbf{v}_{-i}, \nu)$,

[30]Existence of $\hat{v}^*_i(\mathbf{v}_{-i})$ and $\underset{\smile}{v}^*_i(\mathbf{v}_{-i})$ follow from existence of their counterparts near the noiseless limit, which obtain by continuity of equilibrium cutoffs in all parameters for each $\nu > 0$.

[31]The star in Example 2 below satisfies this property. The property can be violated when i is a bridge between two cliques, and with $i = 6$ in Example D1 of Online Appendix D.2.

[32]Observe that regular networks of n agents with $v_j = v$, $\forall j \neq i$ give $\hat{v}^*_i(\mathbf{v}_{-i}) - \underset{\smile}{v}^*_i(\mathbf{v}_{-i}) = n\phi$.

then $c_i^* > c_j^*$, and the likelihoods that i and j place on the other adopting –realizing signals equal to equilibrium cutoffs– revert to equal *one* and *zero*. Therefore, the difference in $\hat{v}_i^*(\mathbf{v}_{-i})$ and $\underset{\sim}{v}_i^*(\mathbf{v}_{-i})$ measures the necessary compensation to i's adoption that offsets the loss in probability (one) placed on each of i's neighbors in $C^*_{m(i)}$ adopting, when moving between the two extremes.

The following example illustrates these properties for the star network.

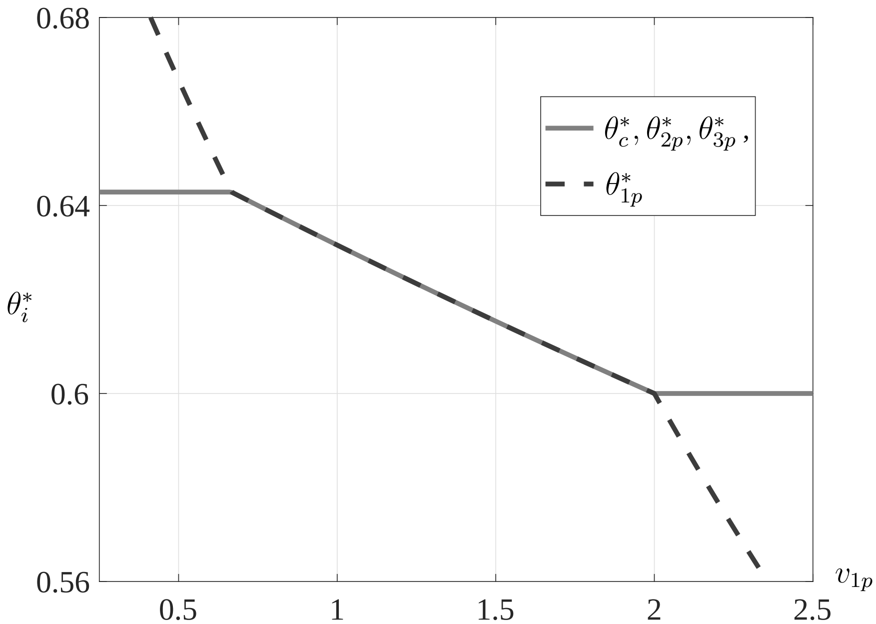

Figure 5: Intrinsic values and local contagion: equilibrium cutoffs in the noiseless limit, versus v_{1p} in the star network.

Example 2. *Take the star network with four agents of Figure 3, Example 1. We take equivalent specifications, but set $v_i = 1$ for $i \neq 1p$, and vary the intrinsic value from adopting of the peripheral agent 1, v_{1p}, over $[0.5, 2.5]$. All calculations are for limit equilibria. We assume the following specification:*[33]

$$u_i(\mathbf{a}_{-i}|\theta) = v_i - 3\frac{(1-\theta)}{\theta} + \sum_{j \in N_i} a_j. \tag{15}$$

Figure 5 plots each agent's equilibrium adoption cutoff, both for agents $c, 2p, 3p$ (solid line) and for agent $1p$ (dashed line). For values of v_{1p} below $\underset{\sim}{v}_{1p}^*(\mathbf{1}) = 2/3$, agent $1p$ lies outside of the coordination set $\{c, 2p, 3p\}$. As v_{1p} rises above $\underset{\sim}{v}_{1p}^*(\mathbf{1})$, agent $1p$ belongs to the same coordination set as the other agents and thus has the same adoption strategy.

[33]Note that $\underline{\theta}, \overline{\theta} \in (0,1)$ obtain for all $v_i, \phi > 0$.

When v_{1p} rises above $\hat{v}^*_{1p}(\mathbf{1}) = 2$, agent $1p$ separates from the others and the agents no longer coordinate together.

One can verify from Figure 5 that all agents coordinate together for each $v_{1p} \in (\underset{\smile}{v}^*_{1p}(\mathbf{1}), \hat{v}^*_{1p}(\mathbf{1}))$. To see this, all agents' cutoffs in the perturbed game converge on the common θ^*_m as $\nu \to 0$, over this range of v_{1p}. As predicted by Proposition 5, expression (14), $\hat{v}^*_i(\mathbf{1}) - \underset{\smile}{v}^*_i(\mathbf{1}) = \frac{4}{4-1}(0+1) = 1.33 = 2 - \frac{2}{3}$.

The above results establish a stark segmentation across coordination sets. This segmentation obtains both in and near the noiseless limit. As one might imagine, this carries implications for comparative statics on the network structure $\mathcal{G}$. Proposition D3 of Online Appendix D.2 derives such comparative statics, with respect to inclusion of additional links into E. The result extends the segmentation to linkage effects, where links between coordination sets *only* impact the cutoff of the coordination set taking a higher cutoff (when the two cutoffs are ordered). Example D1 of that section illustrates this for networks where one agent bridges two cliques.

The next section considers the welfare and policy implications of our model.

8 Welfare and Policy Implications

Proposition 5 establishes a discontinuity in the effects of perturbations to intrinsic values, with agents outside of the perturbed agent's coordination set remaining unresponsive in equilibrium. Proposition 6 reveals an increased robustness in coordination between agents to such perturbations as network effects strengthen. Important questions to any planner remain. In particular, what marginal benefits are realized with adoption subsidies? And, which agents' adoption should be subsidized?

To address these questions, we first develop our welfare analysis near the limit, then quantify the relevant welfare measures as $\nu \to 0$. Consider a policy designer with either of the following two objectives. First, a designer may aim to maximize the *aggregate ex-ante adoption likelihood*. Denote by $H(\cdot)$ the marginal cdf of the state θ and $H'(\cdot)$ its density. Observe that $H(\cdot)$ defines the planner's prior in terms of the distribution of the states of the world θ.[34] As a result, such a designer realizes a marginal increase to this

[34]Note that, as $\nu \to 0$, priors play no role for each agent's adoption decision. See footnote 17 for more discussion.

likelihood from increasing v_i of:

$$ma_i^* := \frac{\partial}{\partial v_i}\sum_{j\in N}\mathbb{E}_{\mathbf{s}}\left[\pi_j^*\right] \xrightarrow{\nu\to 0} \frac{\partial}{\partial v_i}\sum_{j\in N}\mathbb{E}_{\theta}\left[\chi(\theta\geq\theta_j^*)\right],$$

$\chi(\cdot)$ denoting the indicator function. Alternatively, a benevolent planner may aim to maximize the *ex-ante aggregate welfare* across agents. This planner realizes a marginal gain from increasing v_i of:

$$\begin{aligned} mw_i^* \quad &:= \quad \frac{\partial}{\partial v_i}\sum_{j\in N}\mathbb{E}_{s_j}\left[U_j(\boldsymbol{\pi}^*|s_j)\right] \\ &\xrightarrow{\nu\to 0} \quad \frac{\partial}{\partial v_i}\mathbb{E}_{\theta}\left[\sum_{j\in N}\chi(\theta\geq\theta_j^*)\left(v_j+\sigma(\theta)+\phi\sum_{k\in N_j}\chi(\theta\geq\theta_k^*)\right)\right]. \end{aligned}$$

The following obtains.

Proposition 7 (Policy impact)**.** *For each $C_m^*\in\mathcal{C}^*$ and $i\in C_m^*$:*

1.

$$\lim_{\nu\to 0} ma_i^* = \frac{H'(\theta_m^*)}{\sigma'(\theta_m^*)}, \tag{16}$$

2.

$$\lim_{\nu\to 0} mw_i^* = (1-H(\theta_m^*)) + \phi\left(\frac{e(C_m^*, C_m^*)+e(C_m^*)}{|C_m^*|}\right)\frac{H'(\theta_m^*)}{\sigma'(\theta_m^*)}. \tag{17}$$

The main result of this proposition is to show that, whether she maximizes aggregate adoption likelihood or aggregate welfare, the planner needs to target coordination sets and not individuals. Indeed, it is easily verified that expressions (16) and (17) both reduce to functions of coordination-set-level variables. That is, in the noiseless limit, any targeted policy's aggregate impact, on both adoption and ex ante welfare over the network, is left independent of the particular choice of target $i\in C_m^*$, that is, for any i and i' in C_m^*, $\lim_{\nu\to 0} ma_i^* = \lim_{\nu\to 0} ma_{i'}^*$, and $\lim_{\nu\to 0} mw_i^* = \lim_{\nu\to 0} mw_{i'}^*$. Optimal policy design becomes a problem of targeting a given coordination set rather than a particular agent. This contrasts to the literature on games on networks with strategic complementarities, where the planner targets *key players* (Zenou (2016) [56]) to maximize aggregate welfare or adoption. Here, on the contrary, the planner targets *key coordination sets*.

Expression (16) can be interpreted as follows. A subsidy to i's adoption increases adoption amongst C_m^*, while carrying zero influence amongst members of other coordination sets. The effect on the adoption of each member in C_m^* is inversely proportional to $|C_m^*|$ by Proposition 5. Therefore, the aggregate marginal effect of these adoption-based policies is left only as a function of the targeted coordination set C_m^* through the steepness of H (which captures the probability of occurrence of the state) and σ (which captures the elasticity of the value of the technology) at θ_m^*. As a result, the key coordination set to the adoption-maximizing planner will yield the cutoff that maximizes $\frac{H'(\theta_m^*)}{\sigma'(\theta_m^*)}$.[35]

To interpret expression (17), first, the aggregate marginal welfare is now decreasing in θ_m^* through the direct effect on target i's ex-ante welfare, quantified by $(1 - H(\theta_m^*))$. Second, the benevolent planner values the additional externalities between members of the targeted coordination set, as these agents jointly increase their total adoption. Third, subsidies to adoption can generate positive welfare gains to coordination sets that may not contain the target agent i. Precisely, provided an agent j is either in $C_{m(i)}^*$, or takes cutoff $\theta_j^* < \theta_i^*$ and is a neighbor to a member of $C_{m(i)}^*$, j then realizes additional value in all additional state realizations in which her neighbors in $C_{m(i)}^*$ begin to adopt. Each of these components factor into the calculation of mw_i^*, stated in (17).

Substituting $\phi\left(\frac{e(C_m^*, C_m^*)+e(C_m^*)}{|C_m^*|}\right) = -\sigma(\theta_m^*) - \frac{v(C_m^*)}{|C_m^*|}$ into (17), we see that the benevolent planner will broadly target *low coordination sets*, i.e. coordination sets with low cutoff values, but will penalize coordination sets when exhibiting high average valuation $\frac{v(C_m^*)}{|C_m^*|}$. This is precisely because these coordination sets realize high incentive for adoption without providing adequate network externalities to others.[36] Remember that $\mathcal{C}^* := \{C_1^*, \ldots, C_{\bar{m}^*}^*\}$. We have the following result:[37]

Corollary 1 (Key coordination sets)**.** *Assume homogenous intrinsic values, uniform $H(\cdot)$ and $\sigma'(\theta)$ decreasing. Then, the key adoption-maximizing coordination set is $C_{\bar{m}^*}^*$, the highest coordination set, whereas, if v is sufficiently large, the key welfare-maximizing coordination set is C_1^*, the lowest coordination set.*

In other words, when agents are ex ante identical, $H(\cdot)$ is uniform, v large, and $\sigma(\theta)$ satisfies decreasing marginal returns, then maximizing adoption leads to target exactly the opposite coordination set than when maximizing welfare, i.e. respectively, the highest and lowest coordination set.[38] Indeed, when $H'(\theta) = 1$ (uniform distribution), values of

[35]The particular shape of this function, and thus the key coordination set, will depend on the particular application being modeled, and may be arbitrary.

[36]Appendix Section D.3 illustrates targeting under the heterogeneous intrinsic values of Example 2. Indeed, this penalty can alter the welfare-maximizing key coordination set.

[37]When there exists only one coordination set, the targeting problem becomes trivial.

[38]The conditions in Corollary 1 are sufficient, not necessary, and are, therefore, stronger than what we need.

θ are equally likely (i.e., high values of θ are as common as low values of θ) from the planner's perspective, while when $\sigma'(\theta)$ is decreasing, there are decreasing returns of the intrinsic value of state θ. As a result, when maximizing adoption, the planner wants to target the highest coordination set because these agents have the highest cutoff levels and thus the most elastic cutoffs due to decreasing returns. On the other hand, when maximizing welfare, the benevolent planner may want to target the lowest coordination set because it has the highest number of agents and, if v is large enough, it generates the largest network effects.

As an illustration, consider $\sigma(\theta)$ in (15) of Example 2, i.e., $\sigma(\theta) = -3(1-\theta)/\theta$ and uniform $H(\cdot)$. Then, for all $v > -3$, the welfare-maximizing planner targets the lowest coordination set[39] while the adoption-maximizing planner targets the highest coordination set.[40] For example, for the large core-periphery network (d) in Figure 3, the adoption-maximizing planner targets any of $1p, \ldots, 4p$, as these periphery agents' cutoff elasticities are greatest and therefore respond most to the subsidy. The welfare-maximizing planner, on the other hand, targets any member of the core, $1c, \ldots, 6c$, where externalities are most prominent.

In this section, two main messages emerge. First, to maximize aggregate adoption or welfare, one needs to target coordination sets and not individuals, even though the initial aim is to subsidize an individual intrinsic value. Second, these two objectives, aggregate adoption and aggregate welfare, need not lead to the same key coordination set. In particular, the adoption-maximizing planner's optimal target strongly depends on the elasticity of the value of the technology. The welfare-maximizing planner, on the other hand, incorporates expected externalities borne both within the targeted coordination set and across to adjacent coordination sets. As such, the key coordination sets that these two planners identify can be vastly different and, as shown above, depend on the particular shapes of σ and H.

[39] Indeed, in the proof of Corollary 1, we derive a sufficient condition for the welfare-maximizing planner to target the lowest coordination set, i.e., $\frac{\partial}{\partial \theta_m^*} \lim_{\nu \to 0} mw_i^* < 0$. It is given by: $v > \max_{m=1,\ldots,\bar{m}^*} \left\{ \frac{2(\sigma'(\theta_m^*))^2}{\sigma''(\theta_m^*)} - \sigma(\theta_m^*) \right\}$. This condition reduces to $v > -3$ when $\sigma(\theta_m) = -3(1-\theta_m)/\theta_m$.

[40] When $H(\cdot)$ is uniform, for the adoption-maximizing planner to target the highest coordination set, it suffices to show that $\sigma''(\theta) < 0$, which is clearly true when $\sigma(\theta) = -3(1-\theta)/\theta$.

9 Extensions, Variations and Applications

9.1 Extensions and variations

The following extensions of the model are offered. The first two extensions establish that the unique equilibrium selection is broadly robust to the properties of the noise technology of the perturbed game. The subsequent extensions and variation of the model, addressing weighted links, welfare spillovers, and miss-coordination costs (respectively), address the potential for additional/alternative externalities, either non-strategic (in the former) or strategic (in the latter).

Unbounded noise

The above model takes agents' noise supports to be contained within the bounded interval $[-\nu, \nu]$.[41] The positive and normative implications of the model maintain in the noiseless limit under unbounded noise. Consider, for example, the perturbed game where θ is observed with Gaussian noise by all agents: each i observes signal $s_i = \theta + \epsilon_i$, where each $\epsilon_i \sim N(0, \nu)$, $\nu > 0$, and all signals independently drawn conditional on θ. Theorem 1 continues to describe the limit equilibrium $\vec{\boldsymbol{\pi}}$.[42] Therefore, all limiting characterizations, including those of sticky coordination, linkage, and local contagion, as well as the model's welfare properties are intrinsic to the equilibrium selected from the complete information game $G(0)$.

Noise-independent selection

The equilibrium selection in the noiseless limit is not sensitive to the commonality of the noise distribution F. Online Appendix F extends the model setup to establish noise-independent selection (see Frankel et al. (2003) [22], Section 6).

Weighted links

We can extend our results to allow for edge-weights $e_{ij} > 0$ for each $(i, j) \in E$. In this extension, noise-independent selection is maintained (See Online Appendix F). All results

[41] This assumption conveniently yields equilibrium properties near the noiseless limit which are commensurate with the properties of $\vec{\boldsymbol{\pi}}$. In particular, local contagion (Remark 1) and the reach of policy interventions (Proposition 7) extend throughout but remain contained within coordination sets, provided ν is sufficiently small.

[42] An analogous proof to Lemma 1 can be constructed. Beyond this, the theorem's proof is identical.

can be extended upon adjusting for edge-weights. Online Appendix E further discusses this extension.

Spillovers

We can incorporate a spillover function $w_i(\mathbf{a}_{-i}|\theta)$ to augment both $u_i(a_{-i}|\theta)$ and the payoffs to not adopting (now equal to $w_i(\mathbf{a}_{-i}|\theta)$ instead of zero). Under this extension, the equilibrium selected in the limit along with all of the positive results remain. The measure mw_i^* will adjust accordingly to incorporate welfare spillovers, positively and negatively so when $w_i(\mathbf{a}_{-i}|\theta)$ is positive and negative, respectively.

Miscoordination costs

As an application of the model under heterogeneous values, we can set $v_i = v - \phi d_i$ to give:

$$u_i(\mathbf{a}_{-i}|\theta) = v + \sigma(\theta) - \phi \sum_{j \in N_i} (1 - a_j). \tag{18}$$

Such a setup may be construed as homogeneous values under miscoordination costs. In this setting, an inverted analogue of the equilibrium described in Section 6 (under homogenous values) obtains, with more connected coordination sets taking *higher* cutoffs. In equilibrium, agents' links to coordination sets taking lower cutoffs carry zero weight, as these miscoordination costs are avoided with probability one. Links to others within one's coordination set are penalized according to limit likelihoods placed on the neighbors not adopting. And, links to coordination sets taking higher cutoffs are penalized with weights one, with these costs being borne with probability one. Noteworthy, despite this inversion, global coordination on a common cutoff persists within the network families of Proposition 3. Online Appendix G addresses this setup in more detail.

9.2 Applications

Here we map either the basic model or its extensions to the three applications offered in the introduction: Platform adoption, crime, and immigration policy.

Platform and Cryptocurrency Adoption

The adoption of platforms, from new currencies and Blockchain technologies, to online marketplaces and social media platforms, offer natural applications of our model, provided the value to users is increasing in the adoption by neighbors.[43] Take, for example, the adoption by firms to deal in a given cryptocurrency (e.g. Bitcoin).[44] The efficacy of the currency as a medium of exchange is increasing in its adoption by firms that take counterparty positions in business dealings (e.g. suppliers). Each firm i's idiosyncratic value to using the currency can be captured by $v_i + \sigma(\theta)$ (i.e. heterogeneous values), where θ captures the future stability or inflation of the currency. In addition to this value, i realizes a gain due to neighboring counterparty firms' adoption $\phi \sum_{j \in N_i} a_j$.

We can then use our model to derive some interesting results. For example, we can evaluate under which conditions large coordination sets emerge in equilibrium, which, in the context of Bitcoin, would mean that some sectors of the economy are more likely to adopt the currency than others. Also, one can derive policy implications. Bitcoin advocates can identify key coordination sets to subsidize in order to maximize aggregate adoption. If a benevolent planner considers the adoption of the currency to be beneficial to the economy, she can then decide which coordination sets to subsidize in order to maximize aggregate welfare.

Crime

It is well established that delinquency is, to some extent, a group phenomenon, and the source of crime and delinquency is often found in intimate social networks of individuals (see e.g. Sutherland (1947) [51], Warr (2002) [54], Bayer et al. (2009) [8], Dustmann and Piil Damm (2014) [19]). Indeed, delinquents often have friends who have themselves committed several offenses, and social ties among delinquents are seen as a means whereby individuals exert an influence over one another to commit crimes. There are few network models of crime (see e.g. Ballester et al. (2010) [6]) and, to the best of our knowledge, no models that combines both explicit network structure and imperfect information on the probability of being caught in a crime. Let us show how our model captures these different aspects.

Consider a population of potential criminals. Allow $a_i = 1$ to designate agent i's choice to participate in crime. The state of the world θ is unknown and inversely proportional to the presence of police or security, so higher θ means less police.[45] Following Becker (1968)

[43] For products such as software, mobile phones, video game consoles, etc., there are strong peer-effects, which are technological in nature: in order to interact, consumers need to adopt technologies compatible with those of their peers. Network effects are particularly pronounced in product categories with competing technological standards (see e.g., Van den Bulte and Stremersch (2004) [53]).

[44] We thank Ben Golub for suggesting this application.

[45] As in our model, the higher is θ, the more likely someone will adopt, here, commit crime.

[1], we assume that delinquents trade off the costs and benefits of delinquent activities when making their criminal decision. The expected delinquency gains to criminal i are given by:

$$u_i(\mathbf{a}_{-i}|\theta) = \underbrace{\left[1 - p_i\left(\theta, \sum_{j\in N_i} a_j\right)\right] B}_{expected\ benefits} - \underbrace{p_i\left(\theta, \sum_{j\in N_i} a_j\right)\kappa}_{expected\ costs} \tag{19}$$

where $p_i(\theta, \sum_{j\in N_i} a_j)$ is the probability of being caught, B is the benefit of committing crime (i.e. proceeds from crime), and $\kappa > 0$ is the cost of being caught (fine or the cost of being in prison). Quite naturally, $p_i(\theta, \sum_{j\in N_i} a_j)$ is decreasing in θ, as more police increases the probability of being caught, and decreasing in $\sum_{j\in N_i} a_j$, as more connections to criminals increases the technology of committing crime (delinquents improve illegal practice while interacting with their direct delinquent mates) and thus reduces the probability of being caught (see e.g. Sutherland (1947) [51] and Warr (2002) [54]). The utility of not committing crime is normalized to zero.

For simplicity, assume now that $p_i(\theta, \sum_{j\in N_i} a_j) = 1-\rho(\theta)-\tau\sum_{j\in N_i} a_j$, where $\rho'(\theta) > 0$ and $\rho(\overline{\theta}) < 1 - \tau d_{max}$ (which guarantees that $0 < p_i(\theta, \sum_{j\in N_i} a_j) < 1$), where d_{max} is the maximum degree in the network. Then,

$$\begin{aligned} u_i(\mathbf{a}_{-i}|\theta) &= \left(\rho(\theta) + \tau\sum_{j\in N_i} a_j\right) B - \left(1 - \rho(\theta) - \tau\sum_{j\in N_i} a_j\right)\kappa \\ &= \underbrace{-\kappa}_{v} + \underbrace{\rho(\theta)(B+\kappa)}_{\sigma(\theta)} + \underbrace{\tau(B+\kappa)}_{\phi}\sum_{j\in N_i} a_j. \end{aligned}$$

We are in the framework of our model where the incentive to partake in crime is increasing in the criminal activity of direct criminal friends, and decreasing on the presence of police or security.

Compared to the "standard" network crime model with perfect information (Ballester et al. (2010) [6]), the implications are quite different. First, in the latter, each individual commits crime according to her (Bonacich) centrality in the network. Here, viewed through the lens of the results of Section 6, this is not the case. What matters for criminal decision is the coordination set each individual belongs to. This means that our model can explain why some neighborhoods exhibit high crime rates while others do not, even though all individuals are ex ante identical (modulus their network position). Thus, our model offers a different explanation than the one provided in the crime literature (e.g. Glaeser et al. (1996) [26]). Here, the fact that the probability of being caught is unknown, and individuals affect each other's decision to commit crime, lead to coordination (and miscoordination) problems, which result in the endogenous formation of coordination sets, i.e. different pockets of crime.

Second, the policy implications in terms of crime reduction are very different. In the standard literature on criminal networks, the planner wants to target *key players* (Zenou (2016) [56]), i.e. individuals with high intercentrality so that, once removed from the network, they reduce aggregate crime the most. Here, the planner want to target *key coordination sets*, i.e. the parts of the network where crime participation is most influential. This implies that our model is more in favor of place-based rather than individual-based policies as mechanisms to reduce crime.

Immigration Policy

In 2015, more than a million migrants and refugees crossed into Europe. Most of these migrants, who came from the Middle East and Africa, were illegal. Some European countries such as Germany and Sweden were positively inclined towards these migrants whereas other countries, such as Poland and Hungary, were taking strong stance against any possibility of regularizing them.

We can use our framework to model these different immigration policies by allowing $a_i = 0$ to designate the government of country i's choice to take an anti-immigration (i.e. "isolationist") stance.[46] The relative value of taking an inclusive policy ($a_i = 1$), in the form of political support from electorates, is captured by $\sigma(\theta)$. θ may measure a perceived global need for pro-immigration policies, driven by perceptions of foreign conflict or severity of a refugee crisis. We model the inflow of immigrants into country i by $f_i + \sum_{j \in N_i} \tau_{ij}(1 - a_j)$, with $\tau_{ij} > 0$ capturing the overflow of migrants into neighboring country i when $j \in N_i$ takes an anti-immigration stance.[47] The marginal cost to migrant flow is given by $\kappa > 0$. This gives conditional payoff function:

$$\begin{aligned} u_i(\mathbf{a}_{-i}|\theta) &= \sigma(\theta) - \kappa\left(f_i + \sum_{j \in N_i} \tau_{ij}(1 - a_j)\right) \\ &= \underbrace{-\kappa f_i}_{v_i} + \sigma(\theta) - \underbrace{\kappa}_{\phi} \sum_{j \in N_i} \tau_{ij}(1 - a_j). \end{aligned}$$

[46] See Mangin and Zenou (2016) [38] for a first model using global games to study illegal migration.

[47] To interpret, assume density τ_{ij} of migrants can respond to differences across immigration policies in i and j by homing between these neighboring countries. A natural assumption is for τ_{ij} to be proportional to $f_i + f_j$.

This gives a model of miscoordination costs with weighted edges (see Section 9.1). Here, countries in regions with many bordering neighbors are predicted to take anti-immigration stances in more states than countries that are geographically isolated. To avoid the different stances on immigration issues mentioned above, our model suggests that the European Union should have a *common* immigration policy so that all countries belonging to the union could coordinate on a common cut-off strategy. Such a common immigration policy avoids miscoordination costs from excessive migrant flows to pro-immigration countries.

10 Conclusion

This paper studies a coordination model in networks within a global game environment. Agents simultaneously choose to adopt or not. The value of adoption to each agent depends on her idiosyncratic value of adoption, on the quality of the technology, and the number of agents connected to this agent that have adopted the technology. The quality of the technology is not known to agents. Each agent receives a private signal informing her of the technology's quality, and takes a decision. For a range of intermediate signal realizations, the optimal strategy depends on the expectation of other agent's adoption choices. We provide a detailed characterization of the limiting equilibrium in this setting, as information noise diminishes. At the limit, equilibrium strategies are cutoff strategies: each agent adopts if the signal received is above a certain cutoff.

The main contribution of this paper is to provide an algorithm that computes the limiting cutoffs, and to characterize the properties of the cutoffs as a function of the network structure. This characterization allows a partition of the agents into coordination sets, i.e., set of path-connected agents with the same cutoffs. We provide nice properties of these coordination sets. In particular, we show that there is a single coordination set (all players use the same strategies, so they perfectly coordinate) if and only if the network is "balanced", i.e., the average degree of each subnetwork is smaller than the average degree of the full network. Networks that are very symmetric, such as the circle or the complete network, satisfy this property. Surprisingly, very asymmetric networks such as stars also satisfy this property. Importantly, the set of coordination sets is shown to be instrumental to the comparative statics and welfare properties of the model. Contrary to the literature on games on networks, we show that, in order to maximize either aggregate adoption or welfare, the planner needs to target coordination sets and not individuals.

It is left for future work to study the effects of signaling (Angeletos at. al (2006) [4]) or signal jamming (Edmond (2013) [20]) on equilibrium properties such as limit uniqueness and coordination partitioning. Dahleh et al. (2016) [15] study information exchange through a social network, under a symmetric global game; the implications of information transmission under a general network game remains an open question. Equilibrium characterizations under more extensive departures from idiosyncratic noise, such as the introduction of a public signal, also remains for future research.[48]

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A Appendix

In all proofs we use "agents" and "nodes" synonymously.

Proof of Lemma 1. We prove the Lemma assuming $F_i = F$ for each i. However, the result holds for heterogeneous noise distributions by noise-independent selection (see Online Appendix Section F). We first show that for $\nu > 0$ and any pair i, j with cutoffs c_i, c_j:

$$\mathbb{E}[\pi_j(s_j)|s_i = c_i] + \mathbb{E}[\pi_i(s_i)|s_j = c_j] = 1.$$

In particular, when $c_i = c_j = c^*$:

$$\mathbb{E}[\pi_j(s_j)|s_i = c^*] = \frac{1}{2} = \mathbb{E}[\pi_i(s_i)|s_j = c^*].$$

Given $s_i = c_i$, the conditional distribution of θ is $c_i - \nu\epsilon_i$, so:

$$\Pr(c_i - \nu\epsilon_i \leq \theta) = 1 - F\left(\frac{c_i - \theta}{\nu}\right).$$

Moreover, conditional on θ the distribution of s_j is $\theta + \nu\epsilon_j$, so:

$$\mathbb{E}[\pi_j(s_j)|\theta] = \Pr(\theta + \nu\epsilon_j \geq c_j) = 1 - F\left(\frac{c_j - \theta}{\nu}\right).$$

Using the law of iterated expectations:

$$\mathbb{E}[\pi_j(s_j)|s_i = c_i] = \int_\theta \left\{1 - F\left(\frac{c_j - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_i - \theta}{\nu}\right)\right].$$

Similarly:

$$\mathbb{E}[\pi_i(s_i)|s_j = c_j] = \int_\theta \left\{1 - F\left(\frac{c_i - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_j - \theta}{\nu}\right)\right]$$

Taking a sum and using the product rule:

$$\mathbb{E}[\pi_j(s_j)|s_i = c_i] + \mathbb{E}[\pi_i(s_i)|s_j = c_j] = \left\{\left[1 - F\left(\frac{c_i - \theta}{\nu}\right)\right]\left[1 - F\left(\frac{c_j - \theta}{\nu}\right)\right]\right\}_{\theta=-\infty}^{\theta=+\infty}$$

$$= (1 - 0)(1 - 0) - (1 - 1)(1 - 1) = 1.$$

The limiting result (A) follows, since (3) holds for any cutoff and any ν, it continues to hold in the limit as ν goes to zero.

To show (4), recall that:

$$\mathbb{E}[\pi_j(s_j)|s_i = c_i] = \int_\Theta \left\{1 - F\left(\frac{c_j - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_i - \theta}{\nu}\right)\right]$$

We change variable by letting $z = (\theta - c_i)/\nu$, then:

$$\mathbb{E}[\pi_j(s_j)|s_i = c_i] = -\int_\Theta \left\{1 - F\left(\frac{c_j - c_i}{\nu} - z\right)\right\} dF(-z).$$

When $\lim_{\nu\to 0} c_i < \lim_{\nu\to 0} c_j$, for each fixed z:

$$\left\{1 - F\left(\frac{c_j - c_i}{\nu} - z\right)\right\} \to 0, \text{as } \nu \to 0.$$

So by Dominant Convergence Theorem:

$$\lim_{\nu\to 0} \mathbb{E}[\pi_j(s_j)|s_i = c_i] = \int_\theta 0 dF(z) = 0.$$

Similarly we can show: $\lim_{\nu\to 0} \mathbb{E}[\pi_i(s_i)|s_j = c_j] = 1$. $\square$

Proof of Theorem 1. Let us start with the following definition:

Definition 2. *Let K be a closed convex set in $\mathbb{R}^n$. For each $\mathbf{x} \in \mathbb{R}^n$, the* orthogonal projection *(or,* projection*)*[49] *of $\mathbf{x}$ on the set K is the unique point $\mathbf{y} \in K$ such that:*

$$||\mathbf{x} - \mathbf{y}|| \leq ||\mathbf{x} - \mathbf{z}||, \quad \forall \mathbf{z} \in K.$$

We denote $\mathbf{Proj}_K[\mathbf{x}] := \mathbf{y} = \text{argmin}_{\mathbf{z}\in K} ||\mathbf{x} - \mathbf{z}||$.

We can now state the following lemma.

Lemma 2. *The unique vector $\mathbf{q}^*$, the projection of $\mathbf{0}$ onto the $\Phi(\mathcal{W})$, is uniquely characterized by the following two conditions:*

(C1) $\mathbf{q}^* \in \Phi(\mathcal{W})$, *i.e. there exists* $\mathbf{w}^*$ *such that* $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*, \ \forall i$,

[49] See Chapter 1 of Nagurney (1992 [46]) for characterization and properties of this projection operator.

(C2) for any edge $(i,j) \in E$ and for any $z_{ij} \in [0,1]$,

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0.$$

Moreover, we can replace (C2) by the equivalent form:
(C2') $(i,j) \in E, (q_i^* - q_j^*) > 0 \implies w_{ij}^* = 0, w_{ji}^* = 1$.

Proof. We first show necessity. Clearly *(C1)* is just the feasibility condition, hence necessary. For *(C2)*, for any $\mathbf{w}' \in \mathcal{W}$, by optimality of $\mathbf{q}^*$, the following must be true:

$$\eta(t) := ||\Phi((1-t)\mathbf{w}^* + t\mathbf{w}')||^2 \geq ||\Phi(\mathbf{w}^*)||^2 = ||\mathbf{q}^*||^2 = \eta(0)$$

for any $t \in [0,1]$.

Since $\Phi(\cdot)$ is an affine mapping, $\frac{\partial}{\partial t}\Phi((1-t)\mathbf{w}^* + t\mathbf{w}') = \Phi(\mathbf{w}') - \Phi(\mathbf{w}^*)$. Taking the derivative of $\eta(t)$ at $t = 0$, we obtain:

$$0 \leq \eta'(0) = 2\langle \mathbf{q}^*, \Phi(\mathbf{w}') - \Phi(\mathbf{w}^*)\rangle. \tag{A1}$$

Now for any $z'_{ij} \in [0,1]$, we construct a special $\mathbf{w}'$ by only modifying the weights w_{ij}^* and $w_{ji}^* = 1 - w_{ij}^*$ on the edge between i and j in $\mathbf{w}^*$ to $w'_{ij} = z_{ij}$ and $w'_{ji} = 1 - z_{ij}$. Clearly, $\mathbf{w}'$ is still in $\mathcal{W}$. Inequality (A1) becomes:

$$\phi(q_i^*(z_{ij} - w_{ij}^*) + q_j^*(z_{ji} - w_{ji}^*)) \geq 0.$$

However, $z_{ji} - w_{ji}^* = (1 - z_{ij}) - (1 - w_{ij}^*) = -(z_{ij} - w_{ij}^*)$. So the above inequality is equivalent to:

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0.$$

Let us show sufficiency. For any $\mathbf{w}' \in \mathcal{W}$, simple calculation shows that:

$$\langle \mathbf{q}^*, \Phi(\mathbf{w}') - \Phi(\mathbf{w}^*)\rangle = \phi \sum (q_i^* - q_j^*)(w'_{ij} - w_{ij}^*) \geq 0,$$

as each term in the summation is nonnegative. Therefore, $\eta'(0) \geq 0$, moreover $\eta(\cdot)$ is clearly convex in $t \in [0,1]$.[50] Therefore,

$$\eta(1) - \eta(0) \geq (1-0)\eta'(0) \geq 0,$$

[50] As Φ is affine and $||\mathbf{x}||^2$ is a convex function of $\mathbf{x}$

that is:

$$||\Phi(\mathbf{w}')||^2 \geq ||\Phi(\mathbf{w}^*)||^2 = ||\mathbf{q}^*||^2$$

since $\mathbf{w}'$ is arbitrary, and indeed $\mathbf{q}^*$ is the projection of $\mathbf{0}$ onto $\Phi(\mathcal{W})$.

Now we need to verify that for any edge ij with $(i,j) \in E$, *(C2)* is equivalent to *(C2')*:

$$\left\{(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0, \forall z_{ij} \in [0,1]\right\} \Leftrightarrow \left\{(i,j) \in E, (q_i^* - q_j^*) > 0 \Rightarrow w_{ij}^* = 0, w_{ji}^* = 1\right\}.$$

If so, then $q_i^* > q_j^* \implies w_{ij}^* = 0$ and $w_{ji}^* = 1$; $q_i^* < q_j^* \implies w_{ij}^* = 1$ and $w_{ji}^* = 0$.

From *(C2)* to *(C2')*: Suppose $q_i^* > q_j^*$, and let $z_{ij} = 0$. We have $(q_i^* - q_j^*)(0 - w_{ij}^*) \geq 0$, and by $w_{ij}^* \geq 0$ it must be the case that $w_{ij}^* = 0$. Similarly, assuming $q_i^* < q_j^*$ and picking $z_{ij} = 1$ shows that $w_{ij}^* = 1$.

From *(C2')* to *(C2)*: If $q_i^* > q_j^*$ and $w_{ij}^* = 0$, then for any $\forall z_{ij} \in [0,1]$, $(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) = (q_i^* - q_j^*)(z_{ij}) \geq 0$. Similarly, if $q_i^* < q_j^*$ and $w_{ij}^* = 1$, then for any $\forall z_{ij} \in [0,1]$, $(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) = -(q_i^* - q_j^*)(1 - z_{ij}) \geq 0$.

□

Let us now prove the theorem. First, we write down a few necessary conditions for the limiting equilibrium. The cutoffs in the limit must satisfy the indifference conditions:

$$v_i + \sigma(\theta_i^*) + \phi \sum_{j \in N_i} w_{ij}^* = 0, \forall i,$$

where

$$w_{ij}^* = \lim_{\nu \to 0} \mathbb{E}[\pi_j(s_j) | s_i = c_i].$$

Clearly, $w_{ij}^* + w_{ji}^* = 1$ by Lemma 1. Let $q_i^* = -\sigma(\theta_i^*), i \in N$. Then $\theta_i^* < \theta_j^*$ if and only if $q_i^* > q_j^*$. Then $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$, $\forall i$. Moreover, for any connected agent i and j, suppose $\theta_i^* < \theta_j^*$, then $q_i^* > q_j^*$, and $w_{ij}^* = 0$ and $w_{ji}^* = 1$ by Lemma 1.

As a result, $\mathbf{q}^*$ satisfies the two conditions stated in Lemma 2, therefore $\mathbf{q}^*$ must be the projection of $\mathbf{0}$ onto $\Phi(\mathcal{W})$, which proves the theorem. □

Observe that the vector $\mathbf{q}^*$ is a projection of the origin onto the compact, convex space $\Phi(\mathcal{W})$, which is the image of $\mathcal{W}$ under the mapping Φ:

$$\mathbf{q}^* = \mathbf{Proj}_{\Phi(\mathcal{W})}[\mathbf{0}],$$

for $\mathbf{0}$ the vector of zeros in $\mathbb{R}^n$. Denoting $T := \sum_{i \in N} \Phi_i(\mathbf{w})$, and $\mathbf{1}$ the unit vector in

$\mathbb{R}^n$,[51] observe also that, since the set $\Phi(\mathcal{W})$ lies on the hyperplane $H = \{\mathbf{x} \in \mathbf{R}^n, \sum_i x_i = \sum_i v_i + \phi e(N) = T\}$, which includes the diagonal vector $\frac{T}{n}\mathbf{1}$, it does not matter which vector one chooses in the projection provided it is a scaling of $\mathbf{1}$ (i.e. it lies on the diagonal). In particular, $\mathbf{q}^*$ is equivalent to the projection of $\frac{T}{n}\mathbf{1}$ onto the convex set $\Phi(\mathcal{W})$, with $\mathbf{q}^* = \frac{T}{n}\mathbf{1}$ when $\frac{T}{n}\mathbf{1} \in \Phi(\mathcal{W})$.[52]

Proof of Proposition 1. To show expression (8), take weighting matrix $\mathbf{w}^*$. Given $q_i^* = q_j^* = q_m^*$ for each $i, j \in C_m^*$ by definition, it must be that:

$$\begin{aligned}
|C_m^*| q_m^* &= \sum_{i \in C_m^*} \left(v_i + \phi \sum_{j \in N_i} w_{ij}^* \right) \\
&= \sum_{i \in C_m^*} \left(v_i + \phi \left(\sum_{j \in N_i \setminus C_m^*} w_{ij}^* + \sum_{j \in C_m^*} w_{ij}^* \right) \right) \\
&= v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*)),
\end{aligned}$$

the final equality following from Lemma 1. Expression (8) follows. □

Proof of Proposition 4. Take v and ϕ and corresponding $\mathbf{q}^*$ from Theorem 1. For each $v' \neq v$ it must be that $q_i'^* = q_i^* + (v - v')\mathbf{1}$, as $\Phi'(\mathcal{W})$ under v' is:

$$\Phi'(\mathcal{W}) = \{\mathbf{q} + (v - v')\mathbf{1} : \mathbf{q} \in \Phi(\mathcal{W})\}.$$

Thus, $q_i^* = q_j^*$ if and only if $q_i'^* = q_j'^*$: $\mathcal{C}^*$ is independent of v. This also shows that $\mathbf{q}^*$ is affine in v with $\frac{\partial q_i^*}{\partial v} = 1$.

Setting $v = 0$, again take ϕ and corresponding $\mathbf{q}^*$ from Theorem 1. For each positive $\phi' \neq \phi$ it must be that $q_i'^* = \frac{\phi'}{\phi} q_i^*$, as $\Phi'(\mathcal{W})$ under ϕ' is:

$$\Phi'(\mathcal{W}) = \{\frac{\phi'}{\phi}\mathbf{q} : \mathbf{q} \in \Phi(\mathcal{W})\}.$$

Again, $q_i^* = q_j^*$ if and only if $q_i'^* = q_j'^*$: $\mathcal{C}^*$ is independent of ϕ. Again, this shows that $\mathbf{q}^*$ is affine in ϕ. $\mathbf{q}^* = v\mathbf{1} + \phi\mathbf{q}_0^*$ then follows. □

[51] Clearly, for any $\mathbf{w} \in \mathcal{W}$, $\sum_{i \in N} \Phi_i(\mathbf{w}) = \sum_{i \in N} v_i + \phi e(N)$.

[52] The mapping $\Phi(\cdot)$ may not be injective. As the dimension of $\mathcal{W}$ is $e(N)$, the image always lies on the hyperplane H, so the dimension of the image is at most $n - 1$.

Proof of Proposition 2. By Theorem 1, existence of a single coordination set is equivalent to:

$$\frac{T}{n}\mathbf{1} \in \Phi(\mathcal{W}),$$

where $T = \sum_i v_i + \phi e(N)$. This can be re-formulated as a feasibility condition to the following linear programming problem:

$$\begin{aligned} & v_i + \phi \sum_{j \in N_i} w_{ij} = \frac{T}{n},\ \forall i \in N, \\ & w_{ij} \geq 0,\ w_{ij} + w_{ji} = 1,\ \forall (i,j) \in E. \end{aligned}$$

given $v_i = v,\ \forall i$, and $T = \sum v_i + \phi e(N) = nv + \phi e(N)$. So the above system is equivalent to:

$$\begin{aligned} & \sum_{j \in N_i} w_{ij} = \frac{e(N)}{|N|},\ \forall i \in N, \\ & w_{ij} \geq 0,\ w_{ij} + w_{ji} = 1,\ \forall (i,j) \in E. \end{aligned} \tag{A2}$$

To show the necessity, suppose there exists a solution $\mathbf{w}^*$ to system (A2). Then:

$$|S|\frac{e(N)}{|N|} = \sum_{i \in S}(\sum_{j \in N_i} w^*_{ij}) \geq \sum_{i,j \in S:(i,j) \in E} w^*_{ij} = e(S) \cdot (1) = e(S)$$

where the first inequality is trivial, and the second inequality follows from the fact that for each edge with two end nodes i, j both in S, $w^*_{ij} + w^*_{ji} = 1$, there are exactly $e(S)$ such links in the summation.

To show sufficiency, we first re-formulate the above condition as a feasibility condition to a network flow problem, and apply Gale's Demand Theorem (see Gale (1957 [23])). From the original network $G = (N, E)$, we construct a specific bipartite network $\tilde{G} = (V, A)$, where the set of nodes is the union $V = V_1 \cup V_2$ where $V_1 = E$ and $V_2 = N$. The arcs (flow) in $\tilde{G}$ are only from V_1 to V_2. In particular, $f \in E = V_1$ is connected to $i \in N = V_2$ in the bipartite graph $\tilde{G} = (V, A)$, if and only if i is one of the end-points of this edge f in the original network G. Clearly $|V_1| = e(N)$, and $|V_2| = |N|$.

Each vertex $i \in V_2$ is a demand vertex, demanding $d_i = \frac{e(N)}{|N|}$ units of a homogeneous goods. Each vertex in $j \in V_1$ is a supply vertex, supplying $s_j = 1$ unit of the same good. Supply can be shipped to demand nodes only along the arcs A in the constructed bipartite network $\tilde{G}$. Gale's Demand Theorem states that there is a feasible way to match demand and supply if and only if for all $S \subset V_2$:

$$\sum_{i \in S} d_i \leq \sum_{j \in N(S)} s_j,$$

where $N(S)$ is the set of neighbors of vertices in S in $\tilde{G}$. Substituting the values of s_j, d_i yields the following equivalent condition

$$|S|\frac{e(N)}{|N|} \leq |N(S)|, \quad \forall\, \emptyset \subset S \subset V_2.$$

Clearly the above condition holds when S is either empty or the whole set N. For any other case of S, from the construction of $\tilde{G}$, the set $N(S)$ is only the edges in E such that at least one endpoint belongs to S. Therefore:

$$|N(S)| = e(N) - e(S^c)$$

where $S^c = N \backslash S$ is the complement set of S. Recall that:

$$|N| = |S| + |S^c|, e(N) = |N(S)| + e(S^c),$$

It is easy to see that:

$$|S|\frac{e(N)}{|N|} \leq |N(S)| \iff \frac{e(N)}{|N|} \leq \frac{|N(S)|}{|S|} \iff \frac{e(N)}{|N|} \leq \frac{e(N) - e(S^c)}{|N| - |S^c|} \iff \frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}.$$

So the feasibility condition is equivalent to the following:

$$\frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}, \quad \forall\, \emptyset \neq S^c \subset N.$$

Since S is an arbitrary subset of N, and S^c is also arbitrary, the sufficiency direction is proved. This establishes Proposition 2. $\square$

Proof of Proposition 3. For any regular network with degree d, for any non empty subset S, $2\frac{e(S)}{|S|} = \frac{\sum_{i \in S} d_i(S)}{|S|} \leq \frac{\sum_{i \in S} d}{|S|} = d = 2\frac{e(N)}{|N|}$, so regular graph is always balanced, in particular, $q_i^* = q_j^* = v + d\phi/2$ for each $i, j \in N$ and $\mathcal{C}^* = \{N\}$ by Theorem 1.

For trees, there are no cycles, so $e(N) = N - 1$, while for each subset S the resulting subnetwork G_S is still cycle-free. Therefore, the number of edges within S is at most

$|S| - 1$, so $e(S) \leq |S| - 1$, and thus:

$$\frac{e(S)}{|S|} \leq \frac{|S|-1}{|S|} \leq \frac{e(N)}{|N|} = \frac{|N|-1}{|N|}.$$

For regular bipartite networks with two disjoint groups B_1, B_2 with size n_1, n_2, we set $w^*_{ij} = \frac{n_1}{n_1+n_2}$ if $i \in N_1, j \in N_2$, and $w^*_{ij} = \frac{n_2}{n_1+n_2}$ if $i \in N_2, j \in N_1$. Clearly this $\mathbf{w}^*$ is a feasible solution to (A2), with $d_i w^*_{ij} = \frac{e(N)}{n_1+n_2} \in (0,1)$ for each $i \in N$. Therefore by Lemma 2, $q^*_i = q^*_j$ for all $i, j \in N$.

If $\mathcal{G}$ is a network with a unique cycle, then $e(N) = N$. For each subset S, the resulting subnetwork $\mathcal{G}_S$ contains at most one cycle, so the number of edges within S is at most $|S|$, so that $e(S) \leq |S|$, and thus:

$$\frac{e(S)}{|S|} \leq \frac{|S|}{|S|} = 1 = \frac{e(N)}{|N|}$$

When $\mathcal{G}$ contains at most four nodes, all networks with three or fewer nodes contain at most one cycle. The only network structures over four nodes that contain more than one cycle are the circle with a link connecting one non-adjacent pair i and j (two networks) and the complete network. For the former, we can show these networks to have one coordination set with weights: $w^*_{ij} = w^*_{ji} = 1/2$, $w^*_{ki} = w^*_{kj} = 5/8$ and $w^*_{ij} = w^*_{ik} = 3/8$ for each $k \neq i, j$. Each weight is within $(0,1)$ and thus by Lemma 2, $q^*_i = q^*_j = q^*_k$ for each $k \neq i, j$. The complete network with 4 nodes and 6 edges is regular, and clearly has a symmetric equilibrium (i.e. one coordination set). Note, when $N = 5$, there exists a network such that two coordination sets emerges. For example, a core with 4 nodes plus one periphery node having one link to one of the core nodes. $\square$

Proof of Proposition 6. To show (13), denote $\hat{q}^*_j$ and $\underaccent{\check}{q}^*_j$ the limit equilibrium cutoffs of $j \in N$ when $v_i = \hat{v}^*_i$ and $v_i = \underaccent{\check}{v}^*_i$, respectively. $\hat{q}^*_i > \underaccent{\check}{q}^*_i$ with $\hat{q}^*_j \geq \underaccent{\check}{q}^*_j$ for $j \neq i$ given uniqueness of $\boldsymbol{\theta}^*$ and strategic complementarities. For any $\underaccent{\check}{\mathbf{w}}^*$ of Theorem 1 under $v_i = \hat{v}^*_i$, we can find some $\hat{\mathbf{w}}^*$ under $v_i = \underaccent{\check}{v}^*_i$ with $\hat{w}^*_{ij} \leq \underaccent{\check}{w}^*_{ij}$ for each $j \neq i$. Moreover, by construction $\hat{w}^*_{ij} = 0$ and $\underaccent{\check}{w}^*_{ij} = 1$ for each $j \in C^*_m$. At each v_i, q^*_i must satisfy

$q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$. Evaluating v_i at $\hat{v}_i$ and $\utilde{v}_i$, and taking differences gives:

$$\begin{aligned} \hat{v}_i - \utilde{v}_i &= (\hat{q}_i^* - \utilde{q}_i^*) + \phi \sum_{j \in N_i} (\utilde{w}_{ij}^* - \hat{w}_{ij}^*) \\ &= (\hat{q}_i^* - \utilde{q}_i^*) + \phi \left(d_i(C_m^*) + \sum_{j \in N_i \setminus C_m^*} (\utilde{w}_{ij}^* - \hat{w}_{ij}^*) \right) \\ &\geq \phi d_i(C_m^*), \end{aligned}$$

giving inequality (13).

To show equality (14), first by Proposition 1, we can write:

$$q_m^* = \frac{v_i + v(C_m^* \setminus \{i\}) + \phi(e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*|}. \tag{A3}$$

At $v_1 = \hat{v}_i^*$ by Proposition D4 and $w_{ij}^* = 0$ for each $j \in C_m^* \setminus \{i\}$, we have $\hat{q}_m^* = \hat{q}_i^* = \hat{v}_i^* + \phi d_i(\utilde{C}_m^*)$, which by equating with (A3) at $v_i = \hat{v}_i^*$ gives:

$$\hat{v}_i^* = \frac{v(C_m^* \setminus \{i\}) + \phi(-|C_m^*| d_i(\utilde{C}_m^*) + e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*| - 1}. \tag{A4}$$

At $v_1 = \utilde{v}_i^*$, $w_{ij}^* = 1$ for each $j \in C_m^* \setminus \{i\}$, giving $\utilde{q}_m^* = \utilde{q}_i^* = \utilde{v}_i^* + \phi(d_i(\utilde{C}_m^*) + d_i(C_m^*))$, which by equating with (A3) at $v_i = \utilde{v}_i^*$ gives:

$$\utilde{v}_i^* = \frac{v(C_m^* \setminus \{i\}) + \phi(-|C_m^*|(d_i(\utilde{C}_m^*) + d_i(C_m^*)) + e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*| - 1}. \tag{A5}$$

Differencing (A4) and (A5) yields equality (14). □

Proof of Proposition 5.

Lipschitz continuity. Note that $\mathbf{q}^*$ is the projection of $\mathbf{0}$ onto the space $\Phi(\mathcal{W})$:

$$\mathbf{q}^*(\mathbf{v}) = \mathbf{Proj}_{\Phi(W)}[\mathbf{0}].$$

Since Φ depends on $\mathbf{v}$ in a linear way, we let $\mathbf{K} = \Phi(\mathcal{W})$ when $\mathbf{v} = \mathbf{0}$. Then for any $\mathbf{v}$:

$$\Phi(\mathcal{W}) = \mathbf{v} + \mathbf{K}.$$

We can rewrite the projection problem as follows:

$$\mathbf{q}^*(\mathbf{v}) = \arg\min_{\mathbf{z}\in\mathbf{v}+\mathbf{K}} ||\mathbf{z}||^2 = \mathbf{v} + \arg\min_{\mathbf{y}\in\mathbf{K}} ||(-\mathbf{v}) - \mathbf{y}||^2 = \mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]$$

The projection mapping is nonexpansive (see chapter 1 of Nagurney 1992 [46]), i.e:

$$||\mathbf{Proj}_{\mathbf{K}}[\mathbf{x}] - \mathbf{Proj}_{\mathbf{K}}[\mathbf{y}]|| \leq ||\mathbf{x} - \mathbf{y}||, \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}_n.$$

So for any $\mathbf{v}$ and $\mathbf{v}'$, we have

$$||\mathbf{q}^*(\mathbf{v}) - \mathbf{q}^*(\mathbf{v}')|| = ||(\mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]) - (\mathbf{v}' + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}'])||$$

$$\leq ||\mathbf{v} - \mathbf{v}'|| + ||\mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]) - \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}']|| \leq 2||\mathbf{v} - \mathbf{v}'||.$$

Hence, $\mathbf{q}^*(\mathbf{v})$ is Lipschitz continuous in $\mathbf{v}$.

Comparative Statics. By Lipschitz continuity, $\mathbf{q}^*(\mathbf{v})$ is differentiable for almost all $\mathbf{v}$. By Proposition 1, each $q_i^* = q_m^*$ for each $i \in C_m^*$ is given by:

$$q_m^* = \frac{\sum_{i\in C_m^*} v_i + \phi(e(\underline{C}_m^*, C_m^*) + e(C_m^*))}{|C_m^*|}.$$

Note that the terms $e(\underline{C}_m^*, C_m^*)$ and $e(C_m^*)$ are constant holding $\mathcal{C}^*$ constant. For generic $\mathbf{v}$, $\mathcal{C}^*$ is locally constant, hence $e(\underline{C}_m^*, C_m^*)$ and $e(C_m^*)$ do not depend on $\mathbf{v}$ locally. The derivative results follows directly.

Monotonicity. $\partial\mathbf{q}^*/\partial\mathbf{v}$ is nonnegative, so $\mathbf{q}^*$ is monotone in $\mathbf{v}$.

□

Proof of Remark 1. Near the limit ($\nu > 0$), for $k \notin m(i)^*$ with $\theta^*_{m(i)} \neq \theta^*_{m(k)}$, then $s_k^* \notin (c_i^* - \nu, c_i^* + \nu)$ for $\nu > 0$ sufficiently small (i.e. for $\nu \ll |\theta^*_{m(i)} - \theta^*_{m(k)}|/2$), and thus for all $i' \in C^*_{m(i)}$, $a_{i'}$ either equals one or zero (depending on $m' < m$ or $m' > m$, respectively) with probability one conditioning on $s_k = s_k^*$. Because this is true for arbitrary k, it is also true for all members of any $m' \neq m(i)$ (including $m(j)$) for $\nu > 0$ sufficiently small (i.e. for $\nu \ll \min_{m'\neq m(i)} |\theta^*_{m(i)} - \theta^*_{m'}|/2$). Given no atoms of F, this must hold in a neighborhood of s_i^*, which implies $\partial s_j^*/\partial s_i^* = 0$ for all $j \notin m(i)$. If instead $k \notin m(i)^*$ but $\theta^*_{m(i)} = \theta^*_{m(k)}$, by $\partial s_j^*/\partial v_i = 0$ for each $j \notin m(i)^*$ when $\theta^*_{m(i)} \neq \theta^*_{m(j)}$ and by

$C^*_{m(k)}, C^*_{m(j)}$ disjoint by assumption, $\partial s^*_j/\partial s^*_i = 0$ again follows. $\partial s^*_j/\partial s^*_i = 0$ then implies $\partial s^*_j/\partial v_i = 0$.

$\square$

Proof of Proposition 7. To compute the adoption probabilities for each player $j \in C^*_m$, the common cutoff θ^*_m drops by exactly $\frac{1}{\sigma'(\theta^*_m)|C^*_m|}$, and the density of the state θ near θ^*_m is just $H'(\theta^*_m)$. Moreover there are $|C^*_m|$ players in the coordination set containing player i, and thus expression (16) follows.

To compute ex ante welfare, we first note that:

$$\omega_j := \lim_{\nu \to 0} \mathbb{E}_{s_j}[U_j(\pi_j(\boldsymbol{\pi}_{-j})|s_j] = \int_{\theta^*_j}^{+\infty} (v_j + \sigma(\theta) + \phi \sum_{k \in N_j} \chi(\{\theta > \theta^*_j\})))dH(\theta), \tag{A6}$$

for indicator function χ. If $j \in \bar{C}^*_m$ or if $\theta^*_j = \theta^*_i$ with $j \notin C^*_m$, then the impact of v_i on ω_j is zero. This leaves j in C^*_m or $\underline{C}^*_m$. We can write:

$$\omega_j = \int_{\theta^*_j}^{\infty} (v_j + \sigma(\theta))dH(\theta) + \phi \sum_{k \in N_j} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta).$$

Recall that as v_i increases, only θ^*_j for those $j \in C^*_m$ change. If $j \in \underline{C}^*_m$, the cutoff θ_j is not affected by v_i. Moreover $\theta^*_j \leq \theta^*_i$, implying:

$$\begin{aligned} \frac{\partial \omega_j}{\partial v_i} &= \frac{\partial}{\partial v_i}\left(\phi \sum_{k \in N_j} \int_{\max(\theta^*_j, \theta^*_k)}^{+\infty} dH(\theta)\right) \\ &= \phi \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)|C^*_m|} |N_k \cap C^*_m| = \phi \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)|C^*_m|} e(\{k\}, C^*_m). \end{aligned}$$

If instead $j \in C^*_m$ ($j = i$ potentially), by Proposition 5 i and j have the same cutoff θ^*_m in some neighborhood of v_i: $\theta^*_j = \theta^*_m$. Thus we have:

$$\begin{aligned} \frac{\partial \omega_j}{\partial v_i} &= \chi(\{j = i\})(1 - H(\theta^*_m)) + dH(\theta^*_m)(v_j + \sigma(\theta^*_m)) \frac{1}{\sigma'(\theta^*_m)|C^*_m|} \\ &\quad + \frac{\partial}{\partial v_i}\left(\phi \sum_{k \in N_j} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta)\right), \end{aligned}$$

by applying Leibniz integral rule to (A6), and substituting $\frac{1}{\sigma'(\theta^*_m)|C^*_m|} = -\frac{\partial \theta^*_j}{\partial v_i}$. Note that:

$$\begin{aligned}
\sum_{k \in N_j} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta) &= \sum_{k \in N_j \cap \underline{C}^*_m} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta) + \sum_{k \in N_j \cap C^*_m} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta) \\
&\quad + \sum_{k \in N_j \cap \bar{C}^*_m} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta) \\
&= \sum_{k \in N_j \cap \underline{C}^*_m} \int_{\theta^*_m}^{\infty} dH(\theta) + \sum_{k \in N_j \cap C^*_m} \int_{\theta^*_m}^{\infty} dH(\theta) \\
&\quad + \sum_{k \in N_j \cap \bar{C}^*_m} \int_{\theta^*_k}^{\infty} dH(\theta).
\end{aligned}$$

Note that if $k \in N_j \cap \underline{C}^*_m$, $\max(\theta^*_j, \theta^*_k) = \theta^*_j = \theta^*_m$; similarly for other terms. Therefore:

$$\frac{\partial}{\partial v_i} \left(\phi \sum_{k \in N_j} \int_{\max(\theta^*_j, \theta^*_k)}^{\infty} dH(\theta) \right) = \phi \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)|C^*_m|} e(\{j\}, C^*_m \cup \underline{C}^*_m),$$

noting that $\partial \theta^*_k / \partial v_i = 0$ when $k \in N_j \cap \bar{C}^*_m$. Together with the equilibrium condition (recall that $j \in C^*_m$ by assumption):

$$v_j + \sigma(\theta^*_j) + \phi \sum_{k \in N_j} w_{jk} = 0, \Longrightarrow v_j + \sigma(\theta^*_j) = v_j + \sigma(\theta^*_m) = -\phi \sum_{k \in N_j} w_{jk}.$$

We may now simplify:

$$\frac{\partial \omega_j}{\partial v_i} = \chi(k = i)(1 - H(\theta^*_m)) - (\phi \sum_{k \in N_j} w_{jk}) \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)|C^*_m|} + \phi \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)|C^*_m|} e(\{j\}, C^*_m \cup \underline{C}^*_m).$$

Summing over all the agents in both sets, we can obtain an aggregate effect of:

$$\begin{aligned}
\frac{\sum_j \omega_j}{\partial v_i} &= (1 - H(\theta^*_m)) + \phi \left(\frac{Z}{|C^*_m|} \right) \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)} \\
&= (1 - H(\theta^*_m)) + \phi \left(\frac{e(\underline{C}^*_m, C^*_m) + e(C^*_m)}{|C^*_m|} \right) \frac{H'(\theta^*_m)}{\sigma'(\theta^*_m)}.
\end{aligned}$$

It suffices to compute:

$$
\begin{aligned}
Z &:= \sum_{j\in \underline{C}^*_m} e(\{j\}, C^*_m) + \sum_{j\in C^*_m} \Big(e(\{j\}, C^*_m \cup \underline{C}^*_m) - \sum_{k\in N_j} w_{jk} \Big) \\
&= e(\underline{C}^*_m, C^*_m) + (e(\underline{C}^*_m, C^*_m) + 2e(C^*_m)) - (e(\underline{C}^*_m, C^*_m) + e(C^*_m)) \\
&= e(\underline{C}^*_m, C^*_m) + e(C^*_m).
\end{aligned}
$$

To show the second equality (second line), the sum $\sum_{j\in C^*_m} \sum_{k\in N_j} w_{jk}$ can be written:

$$
\sum_{j\in C^*_m} \sum_{k\in N_j} w_{jk} = \sum_{j\in C^*_m} \sum_{k\in N_j\cap \underline{C}^*_m} w_{jk} + \sum_{j\in C^*_m} \sum_{k\in N_j\cap C^*_m} w_{jk} + \sum_{j\in C^*_m} \sum_{k\in N_j\cap \bar{C}^*_m} w_{jk}.
$$

Given $j \in C^*_m$, $w_{jk} = 1$ when $k \in N_j \cap \underline{C}^*_m$, and $w_{jk} = 0$ when $k \in N_j \cap \bar{C}^*_m$, we have: (i) $\sum_{j\in C^*_m} \sum_{k\in N_j\cap \underline{C}^*_m} w_{jk} = e(\underline{C}^*_m, C^*_m)$, (ii) $\sum_{j\in C^*_m} \sum_{k\in N_j\cap \bar{C}^*_m} w_{jk} = 0$, and (iii) $\sum_{j\in C^*_m} \sum_{k\in N_j\cap C^*_m} w_{jk} = e(C^*_m)$, as the limit probabilities on each edge in C^*_m sum to one by Lemma 1. It then follows that $\sum_{j\in C^*_m} \sum_{k\in N_j} w_{jk} = e(\underline{C}^*_m, C^*_m) + e(C^*_m)$. $\square$

Proof of Corollary 1. First, under uniform $H(\cdot)$ and $\sigma'(\theta)$ decreasing, $\lim_{\nu\to 0} ma_i$ is clearly increasing in θ^*_m (see (16)) and, thus, an adoption-maximizing planner will always target the highest coordination set.

Next, let us focus on a welfare-maximizing planner. We derive the exact cutoff $\bar{v}$ such that, under homogenous intrinsic values, uniform $H(\cdot)$ and $\sigma'(\theta)$ decreasing and $v > \bar{v}$, $\lim_{\nu\to 0} mw_i$ is decreasing in θ^*_m (so that a welfare-maximizing planner will always target the lowest coordination set). The condition for $\lim_{\nu\to 0} mw_i$ decreasing becomes (see (17)):

$$
\begin{aligned}
& \frac{\partial}{\partial\theta}\left(1 - \theta - \frac{\sigma(\theta)}{\sigma'(\theta)} - \frac{v}{\sigma'(\theta)}\right) < 0 \\
\Leftrightarrow\quad & -1 - \frac{(\sigma'(\theta))^2 - \sigma''(\theta)\sigma(\theta)}{\sigma'(\theta))^2} + \frac{v}{\sigma'(\theta))^2}\sigma''(\theta) < 0 \\
\Leftrightarrow\quad & \sigma''(\theta)\,[v + \sigma(\theta)] < 2(\sigma'(\theta))^2.
\end{aligned}
$$

With $\sigma''(\theta) < 0$, a sufficient condition for $\lim_{\nu\to 0} mw_i$ decreasing for all θ^*_m is $v \geq \bar{v}$ where:

$$
\bar{v} = \max_{m=1,\dots,\bar{m}^*} \left\{ \frac{2(\sigma'(\theta^*_m))^2}{\sigma''(\theta^*_m)} - \sigma(\theta^*_m) \right\}.
$$

$\square$

Online Appendix for "Coordination on Networks"

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B Appendix: Existence and uniqueness of limit equilibrium

Let us formulate expected payoffs when all neighbors use cutoff strategies. Given $\boldsymbol{\pi}_{-i}$ and conditional on signal realization s_i, i's expected payoff from adopting can be written:

$$\begin{aligned} U_i(\boldsymbol{\pi}_{-i}|s_i) &:= \mathbb{E}_\theta\left[\mathbb{E}_{\mathbf{s}_{-i}}\left[u_i(\mathbf{a}_{-i}|\theta)\,|\boldsymbol{\pi}_{-i},\theta\right]\big|\, s_i\right] \\ &= \mathbb{E}_\theta\left[v_i+\sigma(\theta)+\phi\sum_{j\in N_i} r(\theta,c_j;\nu)\middle|\, s_i\right], \end{aligned} \tag{B1}$$

where the conditional likelihood that $j \in N_i$ adopts is given by:

$$r(\theta,c_j;\nu) := \int_{-1}^{1}\pi_j(\theta+\nu\epsilon_j)f(\epsilon_j)d\epsilon_j = \begin{cases} 0 & \text{if } \theta \le c_j-\nu \\ F\left(\frac{\theta-c_j}{\nu}\right) & \text{if } \theta\in(c_j-\nu,c_j+\nu] \\ 1 & \text{if } \theta > c_j+\nu \end{cases}. \tag{B2}$$

Expression (B1) can then be written:[1]

$$U_i(\boldsymbol{\pi}_{-i}|s_i) = v_i + \int_{-1}^{1}\left(\sigma(s_i-\nu\epsilon_i)+\phi\sum_{j\in N_i} r(s_i-\nu\epsilon_i,c_j;\nu)\right)f(\epsilon_i)d\epsilon_i. \tag{B3}$$

Lemma B1. *A Bayesian Nash Equilibrium $\boldsymbol{\pi}^*$ of $G(\nu)$ in cutoff strategies exists.*

[1] This form uses the assumption of dispersed priors; see footnote 17. The analogous condition with weighting on prior H converges on (B3) as $\nu \to 0$.

Proof. We first show that each agent best responds in $G(\nu)$ to a profile of cutoff strategies via a unique cutoff strategy. With $\sigma(\theta)$ strictly increasing and $r(\theta, s_j; \nu)$ weakly increasing in θ, it is immediate that the integrand in (B3) is strictly increasing in s_i. There must then be a unique signal realization $c_i^* \in (\underline{\theta} - \nu, \overline{\theta} + \nu)$ that solves:

$$U_i(\boldsymbol{\pi}_{-i}|c_i^*) = 0, \tag{B4}$$

with adoption optimal for i if and only if $s_i \geq c_i^*$. By continuity of all payoffs in others' cutoffs, we can applying Brouwer's fixed point theorem giving the result. □

We have shown that there is a unique signal $c_i^* \in (\underline{\theta}-\nu, \overline{\theta}+\nu)$ that solves: $U_i(\boldsymbol{\pi}_{-i}|c_i^*) = 0$, with adoption optimal for i if and only if $s_i \geq c_i^*$. Furthermore, there is a unique limit equilibrium in cutoff-strategies. The next result is straightforward to obtain using Lemma B1 and Theorem 1 in Frankel et al. (2003) [22].

Proposition B1. *There exists an essentially unique strategy profile $\vec{\boldsymbol{\pi}}$, which is in cutoff strategies, such that any $\boldsymbol{\pi}(\cdot;\nu)$ surviving iterative elimination of strictly dominated strategies in $G(\cdot;\nu)$ satisfies $\lim_{\nu\to 0} \boldsymbol{\pi}(\nu) = \vec{\boldsymbol{\pi}}$.*

The unique limit equilibrium $\vec{\boldsymbol{\pi}}$ of $G(0)$ is characterized by $\theta_i^* := \lim_{\nu\to 0} c_i^*$, with each i choosing to [not] adopt when $\theta[<] > \theta_i^*$. With Proposition B1, we are free to study cutoff-strategy equilibria of $G(\nu)$, which must converge on $\vec{\boldsymbol{\pi}}$. $U_i(\boldsymbol{\pi}_{-i}|c_i^*) = 0$ for each $i \in N$ define the system of conditions pinning down such equilibria.

C Appendix: Alternative Characterizations

C.1 Projection mapping and characterization: The case of a dyad network

The following example illustrates the unique projection $\mathbf{q}^*$ for the dyad network.

Example C1. *For dyad with agents 1 and 2, $\mathcal{W} = \{w, 1-w : w \in [0,1]\}$, where $w_{12} = w$ and $w_{21} = 1 - w$, and $\Phi(\mathcal{W}) = \{v_1 + \phi w, v_2 + \phi(1-w) : w \in [0,1]\}$. Figure 6 depicts three cases:* (a) $v_1 - v_2 < -\phi$, (b) $\phi \geq v_1 - v_2 \geq -\phi$, *and* (c) $v_1 - v_2 > \phi$.

When the value gap $|v_1 - v_2| > \phi$ in cases (a) *and* (c), *the projection $\mathbf{q}^*$ obtains a corner of $\Phi(\mathcal{W})$. Precisely, $q_1^* < q_2^*$ and $w = 1$ in case* (a), *and $q_1^* > q_2^*$ and $w = 0$ in case* (c). *In case* (b), *$\Phi(\mathcal{W})$ intersects the diagonal, and thus $q_1^* = q_2^*$, with $w \in (0,1)$ when $\phi > v_1 - v_2 > -\phi$.*

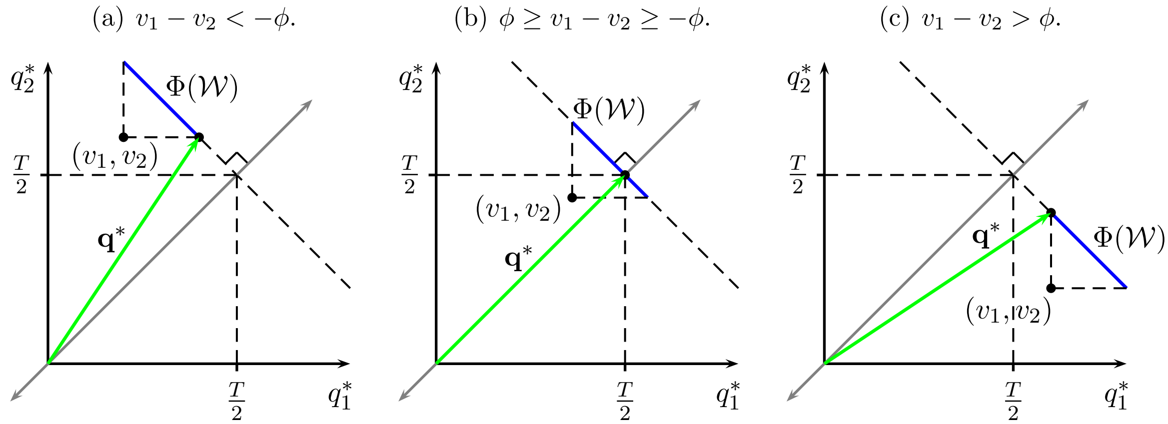

Figure 6: The vector $\mathbf{q}^*$ (green arrow) as the projection of the diagonal (gray arrow) onto $\Phi(\mathcal{W})$ (blue line segment) for the dyad network.

Example C1 shows that $q_1^* = q_2^*$, and thus $\theta_1^* = \theta_2^*$, for a range of value gaps $|v_1 - v_2| \leq \phi$. Provided a sufficient extent of symmetry holds in the limit game $G(0)$, the two agents will take a common cutoff, adopting exactly when the other adopts. Also illustrated with the dyad example, when value gap $|v_1 - v_2| > \phi$ agent i, taking higher limit cutoff places limiting likelihood $w_{ij}^* = 1$ on $j \neq i$ adopting when i observes $\theta = \theta_i^* > \theta_j^*$. Conversely, j places limiting likelihood $w_{ji}^* = 0$ on i adopting when j observes $\theta = \theta_j^*$.

C.2 Algorithmic characterization

Here, we provide an algorithmic approach for calculating limiting coordination sets $\mathcal{C}^*$. The construction of the algorithm is motivated by Proposition D4. With each iteration, an additional limit cutoff is picked-off, starting with θ_1^*.

Algorithm 1 (Combinatorial derivation of coordination sets)**.** *For disjoint agents sets $S, A \subseteq N$, $S \neq \emptyset$, define the function:*

$$\psi(S|A) := \frac{v(S) + \phi(e(S,A) + e(S))}{|S|}.$$

Define $A_0 := \emptyset$. Step $k = 1, \ldots$ of the algorithm is defined as follows:

Step k. *For $A \subset N$, define $\Lambda(A) := \operatorname{argmax}_{\emptyset \neq S \subseteq N \setminus A} \psi(S|A)$. Solve:*

$$B_k = \bigcup_{S \in \Lambda(A_{k-1})} S.$$

Partition B_k into disjoint, connected subsets $\{B_k^1, \ldots, B_k^{p(k)}\}$: $E_{B_k^s \cup B_k^{s'}} = E_{B_k^s} \cup E_{B_k^{s'}}$, $1 \leq s < s' \leq p(k)$. Set $A_k = B_k \cup A_{k-1}$.

Continue until $A_k = N$. Then $\{\{B_1^1, \ldots, B_1^{p(1)}\}, \{B_2^1, \ldots, B_2^{p(2)}\}, \ldots\}$ gives $\mathcal{C}^*$.

To illustrate Algorithm 1, we can apply it to the Large core-periphery network of Example 1. Recall that $C_1^* = \{1c, \ldots, 6c\}$, $C_2^* = \{r\}$, $C_3^* = \{1q, 2q\}$, with each periphery agent jp, $j = 1, \ldots, 4$, giving singleton coordination sets $C_4^*, \ldots, C_7^*$. This yields $B_k = C_k^*$ for $k = 1, 2, 3$, and $B_4 = \cup_{m=4,\ldots,7} C_m^*$. Applying the algorithm to network 2 of Banerjee et al. (2013), illustrated in Subsection 6.2, step two will find $B_2 = C_2^* \cup C_3^*$ (red and blue agents), with each subsequent step $k > 2$ finding coordination set $k+1$, and the algorithm terminates after step five.

The following establishes the relationship between Algorithm 1 and Theorem 1.

Proposition C1 (Duality)**.** *For each step k of Algorithm 1, B_k is itself a solution to $\max_{\emptyset \neq S \subseteq N \setminus A_{k-1}} \psi(S|A_{k-1})$, with $\max_{i \in N \setminus A_{k-1}} q_i^* = \psi(B_k|A_{k-1})$.*

Proof. We prove the first statement by induction on step k. Take $k = 1$. By Theorem 1, the $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$. So, given any nonempty subset $S \subseteq N$:

$$\begin{aligned} \sum_{i \in S} q_i^* &= \sum_{i \in S} (v_i + \phi \sum_{j \in N_i} w_{ij}^*) = \sum_{i \in S} v_i + \phi \sum_{i \in S} \sum_{j \in N_i} w_{ij}^* \\ &= v(S) + \phi \underbrace{\sum_{i \in S} \sum_{j \in N_i \cap S} w_{ij}^*}_{=e(S)} + \phi \underbrace{\sum_{i \in S} \sum_{j \in N_i \cap S^c} w_{ij}^*}_{\geq 0} \geq v(S) + \phi e(S) \end{aligned}$$

Here, $\sum_{i \in S} \sum_{j \in N_i \cap S} w_{ij}^* = e(S)$ as the sum of weights for each link is exactly one, and there are exactly $e(S)$ such links. As a result:

$$\psi(S|\emptyset) := \frac{v(S) + \phi e(S)}{|S|} \leq \frac{\sum_{i \in S} q_i^*}{|S|} \leq \max_{i \in N} q_i^*. \tag{C1}$$

This shows that $\max_{i \in N} q_i^*$ is indeed a upper bound for $\psi(\cdot|\emptyset)$. Next we show that this upper bound is obtained. Define:

$$S^* = \{i \in N : q_i^* = \max_{j \in N} q_j^*\}$$

Then S^* is nonempty. Moreover, for each $i \in S$ and each $j \in N_i \cap (N \setminus S)$, we have

$q_j^* < q_i^* = \max_{k\in N} q_k^*$, as a result, $w_{ij}^* = 0$. So, $\sum_{i\in S}\sum_{j\in N_i\cap(N\backslash S)} w_{ij}^* = 0$. Hence:

$$\psi(S^*|\emptyset) := \frac{v(S^*) + \phi e(S^*)}{|S^*|} = \frac{\sum_{i\in S^*} q_i^*}{|S^*|} = \max_{i\in N} q_i^*.$$

By construction, we also know that S^* is the largest maximizer of $\psi(\cdot|\emptyset)$ (otherwise the last inequality in (C1) will be strict). In fact, S^* is the union of the coordination sets in $\mathcal{C}^*$ such that the cutoff is greatest. We can then partition S^* into disjoint, connected subsets to obtain the limit coordination sets.

As a summary, in the first step of the algorithm, the first cutoff value $\theta_1^* = \sigma^{-1}(-\max_{t\in N} q_t^*)$ is found and the corresponding coordination sets are also found.

The case for each step $k > 1$ is similar; the result follows by induction. Finally, the algorithm must terminate in finite steps, as the set A_k is strictly growing in each step.

To show the second statement of the proposition, the following lemma shows that indeed the largest solution of $\psi(\cdot|A)$ is well-defined.

Lemma C1. *Fixing subset A of N, define function $\psi(\cdot|A)$ from $S \in 2^{N\backslash A}\backslash\{\emptyset\}$ to $\mathbb{R}$ as follows:*

$$\psi(S|A) := \frac{v(S) + \phi(e(S,A) + e(S))}{|S|}.$$

Then if both S' and S'' are maximizer of $\psi(\cdot|A)$, then $S' \cup S''$ is also a maximizer. If $S' \cap S''$ is not empty, then $S' \cap S''$ is also a maximizer of $\psi(\cdot|A)$.

Proof. Let $\beta = \max_{\emptyset\neq S\subseteq N\backslash A} \psi(S|A)$. If both S' and S'' are maximizer of $\psi(\cdot|A)$, then:

$$\begin{aligned}
&v(S') + v(S'') = v(S' \cap S'') + v(S' \cup S''),\\
&e(S', A) + e(S'', A) = e(S' \cap S'', A) + e(S' \cup S'', A),\\
&e(S') + e(S'') \leq e(S' \cap S'') + e(S' \cup S'').
\end{aligned}$$

The first two results direct follow from the definition of $v(\cdot)$, and $e(\cdot, A)$ (recall that S' and S'' are disjointed from A by assumption). The last inequality follows from the observation that:

$$e(S' \cap S'') + e(S' \cup S'') - e(S') - e(S'') = e(S'\backslash S'', S''\backslash S') \geq 0.$$

Assume that $S' \cap S''$ is not empty, then we have the following inequalities:

$$\begin{aligned}
& \psi(S' \cap S''|A)|S' \cap S''| + \psi(S' \cup S''|A)|S' \cup S''| \\
\leq\ & \beta|S' \cap S''| + \beta|S' \cup S''| \\
=\ & \beta|S'| + \beta|S''| \\
=\ & \psi(S'|A)|S'| + \psi(S''|A)|S''| \\
=\ & v(S') + \phi(e(S', A) + e(S')) + v(S'') + \phi(e(S'', A) + e(S'')) \\
\leq\ & v(S' \cap S'') + \phi(e(S' \cap S'', A) + e(S' \cap S'')) \\
& + v(S' \cup S'') + \phi(e(S' \cup S'', A) + e(S' \cup S'')) \\
=\ & \psi(S' \cap S''|A)|S' \cap S''| + \psi(S' \cup S''|A)|S' \cup S''|,
\end{aligned}$$

As a result, all the inequalities are equalities. In particular, $\psi(S' \cup S''|A) = \psi(S' \cap S''|A) = \beta$, i.e., $S' \cap S''$ and $S' \cup S''$ are both maximizers of $\psi(\cdot|A)$.

When $S' \cap S'' = \emptyset$, similarly we can show:

$$\begin{aligned}
& \psi(S' \cup S''|A)|S' \cup S''| \\
\leq\ & \underbrace{\beta|S' \cap S''|}_{=0} + \beta|S' \cup S''| \\
=\ & \beta|S'| + \beta|S''| \\
=\ & \psi(S'|A)|S'| + \psi(S''|A)|S''| \\
=\ & v(S') + \phi(e(S', A) + e(S')) + v(S'') + \phi(e(S'', A) + e(S'')) \\
\leq\ & \underbrace{v(S' \cap S'') + \phi(e(S' \cap S'', A) + e(S' \cap S''))}_{=0} \\
& + v(S' \cup S'') + \phi(e(S' \cup S'', A) + e(S' \cup S'')) \\
=\ & \psi(S' \cup S''|A)|S' \cup S''|,
\end{aligned}$$

which implies that $\psi(S' \cup S''|A)$ is also a maximizer of $\psi(\cdot|A)$. $\square$

The second statement of the proposition now follows from Lemma C1.

$\square$

The proposition shows that each step of Algorithm 1 effectively searches for the maximal set of agents, among all agents left over from prior steps, which yields the greatest collective-average in intrinsic values plus limiting expected network effects. Weights placed on agents taking strictly lower cutoffs (found in earlier steps of the algorithm) are set to one, while the sum of expected weights placed on other members aggregate to the number of links between members (inline with Lemma 1 and Proposition D4).

Theorem 1 and Algorithm 1 can now be viewed as dual problems. The former calculates $\mathbf{q}^*$ yielding the partition $\mathcal{C}^*$ as a bi-product, while the latter constructs $\mathcal{C}^*$ yielding $\mathbf{q}^*$ as a bi-product. Interestingly, Theorem 1 gives the weights w_{ij}^* explicitly in the projection step. Alternatively, in the Algorithm 1, these weights are set either to 0 or 1 for agents in different coordination sets by construction, while they are implicitly implied by Gale's Demand Theorem for agents residing within the same coordination set.

D Appendix: Additional Results

D.1 Bounding limit cutoffs and the largest coordination set

Proposition D4 provides an exact calculation of each θ_m^* as a function of average degrees across all members of C_m^*. The following result provides bounds on limit cutoffs using only the minimal degree within a given agent set. Denote $q_d^{r*} := v + \phi d/2$ and $\theta_d^{r*} := \sigma^{-1}(-q_d^{r*})$ for any regular network of degree d.

Proposition D1 (bounding limit cutoffs)**.**

1. *For each agent set* $S \subseteq N$, $\max_{i \in S} \theta_i^* \leq \theta_d^{r*}$, *setting* $d = \min_{i \in S} d_i(S)$.
2. *For each coordination set* $C_m^* \in \mathcal{C}^*$, $\theta_m^* \geq \theta_{2d}^{r*}$, *setting* $d = \min_{i \in C_m^*} d_i(\underline{C}_m^* \cup C_m^*)$.

Proof. For part *1.*, $c_i^* \leq c_j^*$ for all $i \in S$ and j in regular network $\mathcal{G}$ of degree d follows from supermodularity of $G(\nu)$, uniqueness of c^* for ν small, and $d_i \geq d$ for each $i \in S$. By continuity, $\max_{i \in S} \theta_i^* \leq \theta_d^{r*}$.

For part *2.*, take coordination set C_m^* and $\underline{i} \in \operatorname{argmin}_{i \in C_m^*} d_i(\underline{C}_m^* \cup C_m^*)$. $\underline{i}$'s expected network effect in $G(\nu)$ is no greater than $d = d_{\underline{i}}(\underline{C}_m^* \cup C_m^*)$, which equals expected network effect to each k in a regular network of degree $2d$. Thus, $s_{\underline{i}}^* \geq s_k^*$ for all $\nu > 0$ small. By continuity, $\theta_{\underline{i}}^* \geq \theta_{2d}^{r*}$. □

To illustrate Proposition D1, we return to Example 1 under $v = 0$, $\phi = 1$ to yield $q_d^{r*} = d/2$. As observed in Figure 1, and consistent with part 1 of the proposition, the star and triad-core-periphery networks exhibit a common q_1^* positioned weakly above those of the dyad and triad, $q_1^{r*} = 0.5$ and $q_2^{r*} = 1$, respectively. Likewise, the cores of the quad and large core-periphery networks exhibit q_1^* positioned weakly above $q_3^{r*} = 1.5$

and $q_5^{r*} = 2.5$, respectively. For part 2, the peripheral agents of the star and triad-core-periphery networks carry one link within their coordination sets. All members of these networks exhibit q_1^* at or below q_2^{r*}.

The following applies Propositions D4 and D1 to tree and regular-bipartite networks.

Remark 2 (Bounding limit cutoffs: trees and regular-bipartite networks)**.**

1. *For any tree network,* $\theta_1^{r*} \geq \theta_1^* = \sigma^{-1}(-(v + \phi\frac{|N|-1}{|N|})) \geq \theta_2^{r*}$.
2. *For any regular-bipartite network,* $\theta_{\min\{d_1,d_2\}}^{r*} \geq \theta_1^* = \sigma^{-1}(-(v+\phi\frac{e(N)}{n_1+n_2})) \geq \theta_{2\min\{d_1,d_2\}}^{r*}$.

We see that the limit cutoffs of the dyad and triad bound any tree's limit cutoff from above and below. The common limit cutoff of any regular-bipartite network can also be bounded, both above and below, now by the degree of the network's less-connected side.

The results of Section 6 take non-singleton coordination sets as cases of interest. The next result shows that under homogeneous intrinsic values, $\mathcal{C}^*$ must always exhibit such coordination. Moreover, C_1^* must contain at least four members if $|N| \geq 4$.

Proposition D2. *For homogeneous valuations and any $\mathcal{G}$, there exists at least one coordination set with size at least 4.*

Proof. Assume $|N| \geq 4$. If C_1^* has at most three members, then $e(C_1^*) \leq 1$, with equality when C_1^* gives the complete triad, in which case $q_1^* = v + \phi$ from Proposition 1. Because $\mathcal{G}$ is connected, there must be as least one $j \in \cup_{i \in C_1^*} N_i$. By Proposition 1, $q_j^* \geq v + \phi$, implying that j either coordinates with C_1^* or $q_j^* > q_1^*$, either case giving a contradiction. □

D.2 Linkage

Here we consider a comparative static with respect to the network structure $\mathcal{G}$. Consider network $\mathcal{G}_{+ij}$, defined as the supergraph of $\mathcal{G}$ which includes the additional link ij, and $\mathcal{C}_{+ij}^*$ the limit partition under $\mathcal{G}_{+ij}$. While adding links can affect the limit partition, Proposition D4 can be employed to verify when the limiting coordination is left unchanged: for $\mathcal{C}_{+ij}^* = \mathcal{C}^*$. For these cases, Proposition D3 establishes a disparity in the effects of included links on equilibrium cutoffs. While additional links unambiguously encourage adoption amongst agents taking higher cutoffs, the equilibrium adoption of the agent taking a lower cutoff may not be influenced by the additional link. For the following, and in the sequel, we focus on changes to $\mathbf{q}^*$, again noting the one-to-one correspondence with $\boldsymbol{\theta}^*$ via (6). Let $q_{m,+ij}^*$ correspond to coordination set C_m^* under network $\mathcal{G}_{+ij}$.

Proposition D3 (linkage: limit cutoffs)**.** *Take i, j with $m(i) \geq m(j)$, $ij \notin E$, such that $\mathcal{C}^*_{+ij} = \mathcal{C}^*$. If:*

1. *$\theta^*_{m(i)} > \theta^*_{m(j)}$, then:*

$$q^*_{m(i),+ij} - q^*_{m(i)} = \phi \frac{1}{|C^*_{m(i)}|}, \qquad \text{and} \qquad q^*_{m(j),+ij} - q^*_{m(j)} = 0;$$

2. *$m(i) = m(j) =: m$, then:*

$$q^*_{m,+ij} - q^*_m = \phi \frac{1}{|C^*_{m(i)}|}.$$

Proof. Given $\mathcal{C}^*_{+ij} = \mathcal{C}^*$, C^*_m is unchanged upon inclusion of link (i, j). Moreover, if $\theta^*_{m(i)} > \theta^*_{m(j)}$, then this ordering must maintain upon inclusion of (i, j), else contradicting $\mathcal{C}^*_{+ij} = \mathcal{C}^*$. We may directly apply Proposition 1:

$$\begin{aligned}
q^*_{m(i)} &= \frac{v(C^*_m) + \phi(e(C^*_m, \underline{C}^*_m) + e(C^*_m))}{|C^*_m|}, \\
q^*_{m(i),+ij} &= \frac{v(C^*_m) + \phi(e(C^*_m, \underline{C}^*_m) + e(C^*_m) + 1)}{|C^*_m|}, \\
q^*_{m(j),+ij} &= q^*_{m(j)} \quad if\ \theta^*_{m(i)} > \theta^*_{m(j)},
\end{aligned}$$

the second equality holding whether $j \notin m(i)$ with $\theta^*_{m(i)} > \theta^*_{m(j)}$(for *1.*) or $j \in m(i)$ (for *2.*). Differencing $q^*_{m(j),+ij}$ and $q^*_{m(j)}$ gives the result. □

The inclusion of links between members of distinct coordination sets will expand adoption outcomes within the coordination set taking higher cutoff, but carry zero influence on adoption within the coordination set taking lower cutoff. While the inclusion of links between members of the same coordination set directly influences the two members' incentives to adopt, the expansion in adoption outcomes within the coordination set is comparable to that resulting from a single link to an agent taking a lower cutoff.

Example D1. *Consider network structures of the form depicted in Figure 7, under homogenous values $v_i = v$ for each $i \in N$. Agents 1 through 5 and 7 through 10 form cliques, with agent 6 bridging the two cliques with varying connectivity to each clique. We denote ℓ_1 the number of links that 6 has with agents in $\{1, \ldots, 5\}$, and ℓ_2 the number of links that 6 has with agents in $\{7, \ldots, 10\}$. Table 1 summarizes the equilibrium coordination sets, and provides $\hat{\mathbf{q}}^*$ from Theorem 1 for various values of (ℓ_1, ℓ_2).*

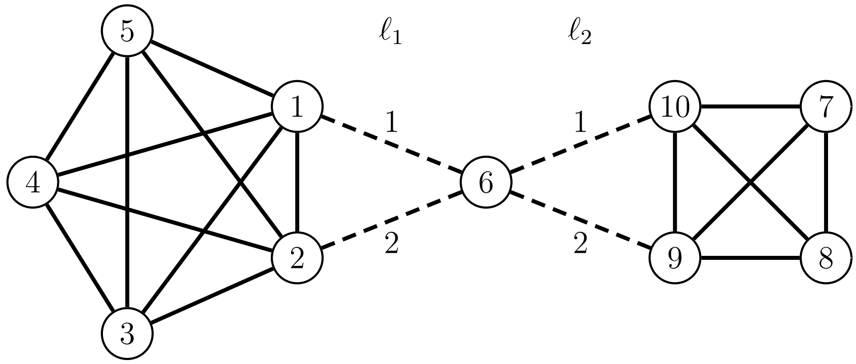

Figure 7: Coordination and bridging.

(ℓ_1, ℓ_2)	$\mathcal{C}^*$	$\hat{\mathbf{q}}^*$
$(0,0)$		$(2, 1.5, 0)$
$(0,1)$	$\{\{1,\ldots,5\},\{7,\ldots,10\},\{6\}\}$	$(2, 1.5, 1)$
$(1,0)$		$(2, 1.5, 1)$
$(1,1)$		$(2, 1.6)$
$(0,2)$	$\{\{1,\ldots,5\},\{6,\ldots,10\}\}$	$(2, 1.6)$
$(1,2)$		$(2, 1.8)$
$(2,0)$	$\{\{1,\ldots,6\},\{7,\ldots,10\}\}$	$(2, 1.5)$
$(2,1)$		$(2, 1.75)$
$(2,2)$	$\{N\}$	(2)

Table 1: Coordination sets $\mathcal{C}^*$ and $\hat{\mathbf{q}}^*$ for agent 6 linkage.

As agent 6 forms two links with each of the two cliques, all of the agents coordinate together on a common cutoff in the noiseless limit. While the total number of links that 6 carries with each clique lies strictly below that of the members of each respective clique, 6 functions as a coordination bridge, synchronizing adoption strategies through the economy. When the number of links to either clique drops below two, 6 either coordinates with one of the two cliques, or coordinates with neither when holding only one link. We see that forming one link with either clique increases $\hat{q}_6^*$ *by exactly* $1 = 1/|\{6\}|$*, while having no impact on cutoffs of the clique, as predicted by Proposition D3 part 1.*[2] *When agent 6 holds one link with clique* $\{7, \ldots, 10\}$ *and adds an additional link to the clique, we see an*

[2]Likewise, if agent 6 holds two links with clique $\{1, \ldots, 5\}$ and adds a link to clique $\{7, \ldots, 10\}$, we see an increase in $\hat{q}_i^*$, $i = 7, \ldots, 10$ of $0.25 = 1/|\{7, \ldots, 10\}|$, specifically from 1.5 to 1.75, as predicted by Proposition D3, part 1.

increase in $\hat{q}_i^*$, $i = 6, \ldots, 10$, of $0.2 = 1/|\{6, \ldots, 10\}|$, *that is from* 1.6 *to* 1.8, *as predicted by Proposition D3 part 2.*

D.3 Characterizations with Heterogeneous Intrinsic Valuations

The next proposition extends Proposition 2 to heterogeneous valuations. It result provides a partial characterization of when a set of agents C_m^* coordinate together: it takes as given the coordination by all others agents $N \backslash C_m^*$ on higher or lower cutoffs. Denote $\underline{C}_m^*$ as above and $\bar{C}_m^* := \cup_{m'<m} C_{m'}^*$. Again, $v(S) := \sum_{i \in S} v_i$ for $S \subseteq N$.

Proposition D4 (Common coordination)**.** *Conditional on $\underline{C}_m^*$ adopting and $\bar{C}_m^*$ not adopting with probability one, connected agent-set $C_m^* = N \backslash (\underline{C}_m^* \cup \bar{C}_m^*)$ coordinate together* if and only if *for every nonempty $S \subset C_m^*$:*

$$\frac{v(S) + \phi(e(S, \underline{C}_m^*) + e(S))}{|S|} \leq \frac{v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))}{|C_m^*|}. \tag{D1}$$

Proof. We can extend the proof of Proposition 2, as follows. To show condition (D1), the proof is analogous provided we modify the values of demand d_j and supply s_i, accounting for v_i, links between C_m^* and $\underline{C}_m^*$ and constraining to subgraph $\mathcal{G}_{C_m^*}$. For this, define $\tilde{V}_1 = E_{C_m^*}$ and $\tilde{V}_2 = C_m^*$. Define:

$$\tilde{s}_j = \phi,\ \forall j \in \tilde{V}_1, \text{ and } \tilde{d}_i = \frac{v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))}{|C_m^*|} - (v_i + \phi d_i(\underline{C}_m^*)),\ \forall i \in \tilde{V}_2,$$

It is straight forward to check that:

$$\sum_{j \in \tilde{V}_1} \tilde{s}_j = \phi e(C_m^*) = \sum_{i \in \tilde{V}_2} \tilde{d}_i.$$

The result just follows from Gale's Demand Theorem.

□

The condition (D1) must hold for all subsets of C_m^*. That is, a violation by some $S \subset C_m^*$ implies that S carries a strictly greater incentive for adoption than all others in C_m^* when they take a higher limit cutoff. Note that setting $\underline{C}_m^* = \bar{C}_m^* = \emptyset$, Proposition D4 reduces to Proposition 2.

By setting $\underline{C}_m^* = \bar{C}_m^* = \emptyset$ in Proposition D4, we can establish the analogue to Proposition 2 under heterogeneous values.

Corollary D1. *Under heterogeneous valuations, a single coordination set exists (i.e.* $\mathcal{C}^* = \{C_1\}$*) if and only if for every nonempty* $S \subset N$*,*

$$\frac{v(S) + \phi e(S)}{|S|} \leq \frac{v(N) + \phi e(N)}{|N|}. \tag{D2}$$

When this condition is satisfied, the common cutoff is $\theta_1^* = \sigma^{-1}(-\frac{v(N)+\phi e(N)}{|N|})$. Condition (D2), holding for all $\emptyset \neq S \subset N$, gives the extension of network balance to heterogeneous valuations. Moreover, we see that Algorithm 1 terminates in one step *if and only if* there exists a unique coordination set. That is, when N is the maximizer of $\psi(\cdot|\emptyset)$:

$$\psi(S|\emptyset) \leq \psi(N|\emptyset), \quad \forall\, \emptyset \neq S \subset N.$$

In other words,

$$\frac{v(S) + \phi e(S)}{|S|} \leq \frac{v(N) + \phi e(N)}{|N|}, \quad \forall\, \emptyset \neq S \subset N,$$

which is exactly the condition identified in equation (D2).

We next give the proof of Remark 1.

Proof of Remark 1. Near the limit ($\nu > 0$), for $k \notin m(i)^*$ with $\theta^*_{m(i)} \neq \theta^*_{m(k)}$, then $s_k^* \notin (s_i^* - \nu, s_i^* + \nu)$ for $\nu > 0$ sufficiently small (i.e. for $\nu \ll |\theta^*_{m(i)} - \theta^*_{m(k)}|/2$), and thus for all $i' \in C^*_{m(i)}$, $a_{i'}$ either equals one or zero (depending on $m' < m$ or $m' > m$, respectively) with probability one conditioning on $s_k = s_k^*$. Because this is true for arbitrary k, it is also true for all members of any $m' \neq m(i)$ (including $m(j)$) for $\nu > 0$ sufficiently small (i.e. for $\nu \ll \min_{m' \neq m(i)} |\theta^*_{m(i)} - \theta^*_{m'}|/2$). Given no atoms of F, this must hold in a neighborhood of s_i^*, which implies $\partial s_j^*/\partial s_i^* = 0$ for all $j \notin m(i)$. If instead $k \notin m(i)^*$ but $\theta^*_{m(i)} = \theta^*_{m(k)}$, by $\partial s_j^*/\partial v_i = 0$ for each $j \notin m(i)^*$ when $\theta^*_{m(i)} \neq \theta^*_{m(j)}$ and by $C^*_{m(k)}, C^*_{m(j)}$ disjoint by assumption, $\partial s_j^*/\partial s_i^* = 0$ again follows. $\partial s_j^*/\partial s_i^* = 0$ then implies $\partial s_j^*/\partial v_i = 0$.

□

We close this section exploring optimal targeting in Example 2. Assume H is uniform with support $[0, 1]$: $H'(\theta) = 1$. By Corollary 1, the adoption maximizing key coordination set is $\bar{m}^*$. With payoff function (15) it is straight forward to calculate:

$$\lim_{\nu \to 0} m w_i^* = 1 + 2\theta^*_{m(i)} - (\theta^*_{m(i)})^2 \left(3 + \frac{v(C^*_{m(i)})}{|C^*_{m(i)}|}\right).$$

As v_{1p} approaches $\hat{v}^*_{1p}(\mathbf{1}) = 2$, $\lim_{\nu\to 0} mw^*_i$ converges on 0.67, for each $i \in C^*_1 = N$; per the above, the welfare-maximizing planner is indifferent over which agent to target. At $\hat{v}^*_{1p}(\mathbf{1}) = 2$, however, the limit partition discontinuously shifts to become $\{\{1p\}, \{c, 2p, 3p\}\}$, $\lim_{\nu\to 0} mw^*_{1p}$ drops to 0.4, and $\lim_{\nu\to 0} mw^*_i$, $i = c, 2p, 3p$ rises to 0.76. Moreover, one can verify $\lim_{\nu\to 0} mw^*_{1p} < \lim_{\nu\to 0} mw^*_i$ (for $i = c, 2p, 3p$) for all $2 < v_{1p} < 9$.[3] To interpret, the welfare-maximizing planner optimally subsidizes the remaining coordination set $C^*_2 = \{c, 2p, 3p\}$ to move the four agents closer together.[4] Doing so capitalizes on contagion within C^*_2 and externalities from C^*_2 to $1p$. Not until θ^*_{1p} is sufficiently below θ^*_2 will the gain from $1p$'s (individual) marginal adoption outweigh the additional externalities captured through targeting C^*_2.

Notice that as v_{1p} rises above 2, the welfare-maximizing key coordination set becomes C^*_2. We see that the benevolent planner effectively penalizes $1p$ for holding large $v_{1p} > v_i$, $i = c, 2p, 3p$, and opts to target the coordination set with the second lowest cutoff.

E Weighted links

First, we write i's ex-post payoff as follows:

$$u_i(\mathbf{a}_{-i}|\theta) = v_i + \sigma(\theta) + \phi \sum_{j\in N_i} e_{ij} a_j. \tag{E1}$$

Theorem 1 remains unchanged.[5] We extend the following definitions to allow for weighted links. First, now define i's *weighted degree* $d_i(S) \sum_{j\in N_i \cap S} e_{ij}$. The definitions of $e(\cdot,\cdot)$ and $e(\cdot)$ then go through:

$$e(S, S') = \sum_{i\in S} d_i(S'),$$

$$e(S) = \frac{1}{2} \sum_{i\in S} d_i(S).$$

Propositions 1 solving for equilibrium q^*_m and Proposition 10 characterizing global coordination remain unchanged.[6] Proposition 3 is reserved for unweighted graphs, as the following three-agent counterexample can easily be constructed. If $e_{ij} \gg e_{jk}$ with $e_{ik} = 0$,

[3] For this, it is easy to verify $\theta^*_{1p} = 3/(3 + v_{1p})$ when $v_{1p} > 2$ and $\theta^*_{1p} < \theta^*_i$, $i = c, 2p, 3p$.

[4] In this case, $w^*_{cj} = 1$ and $w^*_{jc} = 0$ when $v_{1p} > 2$, and so the planner will want to target either $2p$ or $3p$ to ensure the three agents continue to coordinate together. However, this is a construct of $\hat{v}^*_c((\mathbf{1}_{-1p}, 2)) = 1$, a case that does not hold generically.

[5] The proof of Theorem 1 requires only modest adjustments; we leave this for the reader.

[6] Again, these require modest adjustments to proofs.

then agents i and j will coordinate together, and agent k will take a strictly higher cutoff. Corollary 4, establishing that $\mathcal{C}^*$ is independent of ϕ, will continue to hold. And when intrinsic values are heterogeneous, the comparative statics results Propositions 5 and 6 also go through, provided we apply our updated definition of $d_i(\cdot)$ for Proposition 6.

Importantly, noise independent selection (Section F) continues to hold when links are weighted and symmetric. The relevant potential function becomes:

$$P(\mathbf{a}|\theta) := \sum_{i \in N} (v_i + \sigma(\theta)) a_i + \frac{1}{2}\phi \sum_{i,j \in N; i \neq j} e_{ij} a_i a_j, \text{ where } \mathbf{a} \in \{0,1\}^N, \tag{E2}$$

and the results of Frankel et al. (2003 [22]), Oyama and Takahashi (2017 [48]) and Basteck et al. (2013 [7]) carry through.

F Noise-independent selection

Here we consider the robustness of equilibrium selection to heterogeneous noise structures. Consider the following extension.[7]

Information Structure. In the perturbed game, each i realizes signal $s_i = \theta + \nu\epsilon_i$, $\nu > 0$, where ϵ_i is distributed via density function f_i and cumulative function F_i with support within $[-1, 1]$. Signals are independently drawn across agents conditional on θ.

As shown in Theorem 1, the limiting cutoff θ_i^* are fully determined by the parameters $\mathbf{v}, \phi, \sigma(\cdot)$, and $\mathcal{G}$, in particular, the cutoffs are independent of the noise distribution F. In this appendix, we provide an alternative proof of the noise-independent selection result from a potential game approach.

In the simple case with two-player and binary action coordination game (dyad case in our paper), as shown in Carlsson and van Damme (1993) [13], the risk-dominant equilibrium is selected by global game and it is independent of noise distribution. Frankel et al. (2003) [22] generalize this result to n-player supermodular games which yield a potential, which applies to our setting under arbitrary network structures. Recall that in our coordination game, each player has a binary action $a_i \in \{0, 1\}$. Define the following function:

$$P(\mathbf{a}|\theta) := \sum_{i \in N} (v_i + \sigma(\theta)) a_i + \frac{1}{2}\phi \sum_{i,j \in N; i \neq j} a_i a_j, \text{ where } \mathbf{a} \in \{0,1\}^N. \tag{F1}$$

[7] Frankel et al. (2003) [22] Section 6 addresses such an enrichment.

It is straightforward to check that $P(\mathbf{a}|\theta)$ is a potential function of game $G(0)$ at θ (Monderer and Shapley 1998 [39]), by the following

$$\begin{aligned} P(a_i', \mathbf{a}_{-i}|\theta) - P(a_i, \mathbf{a}_{-i}|\theta) &= (a_i' - a_i)\left(v_i + \sigma(\theta) + \phi \sum_{j \in N_i} a_j\right) \\ &= u_i(a_i', \mathbf{a}_{-i}|\theta) - u_i(a_i, \mathbf{a}_{-i}|\theta). \end{aligned}$$

Moreover, the potential P is supermodular in $(a_i, \mathbf{a}_{-i})$ for fixed θ, and strictly supermodular in (a_i, θ) for fixed $\mathbf{a}_{-i}$. As a result, by Frankel et al. (2003 [22]), Oyama and Takahashi (2017 [48]) and Basteck et al. (2013 [7]), the game $G(0)$ has an exact potential, therefore the maximizer of the potential is selected by the global game, and this selection is independent of noise distribution F.[8]

The connection between the potential game approach and our approach in Theorem 1 can be understood from the following relationship:[9] for generic $\mathbf{v}$,

$$\theta_i^* = \inf\{\theta \in \Theta | \exists \mathbf{a}_{-i} \text{ such that } (1, \mathbf{a}_{-i}) \in \arg\max_{\mathbf{a}} P(\mathbf{a}|\theta)\}.$$

While the potential approach requires solving the maximization of P for each θ, which makes it challenging for comparative statics due to discreteness of $\mathbf{a}$, our approach has the advantage that more precise information about the equilibrium cutoff points θ_i^* is obtained using Theorem 1 and the projection algorithm. Moreover, the information coordination set, i.e., who coordinates with whom, is also directly decoded using the cutoff values, which enables us to conduct comparative statics with respect to network structure and valuations in a much simpler manner.

G Miscoordination costs

When a single coordination set obtains the common cutoff is $\theta_1^* = \sigma^{-1}(-v + \phi\frac{e(N)}{|N|})$, by Proposition D4. Moreover, one can apply Proposition D4 to reconstruct Proposition 2 as the equivalent condition for a single coordination set. To show this, set $N_1 = \emptyset$ with

[8] Moreover, Ui (2001 [52]) shows that the selected equilibrium is *robust* in the sense of Kajii and Morris (1997 [36]). See Morris and Ui (2005 [43]), Oyama and Takahashi (2017 [48]) for further discussions.

[9] Note that for generic $\mathbf{v}$, the potential P has a unique maximizer.

$v_i = v - d_i$ in Proposition D4 to obtain:

$$\frac{|S|v - \sum_{i \in S} d_i + e(S)}{|S|} \leq \frac{|N|v - \sum_{i \in N} d_i + e(N)}{|N|}, \quad \forall \emptyset \neq S \subset N,$$

which, given $-\sum_{i \in S} d_i + e(S) = -e(S, S^c) - e(S)$, is equivalent to:

$$\frac{e(S, S^c) + e(S)}{|S|} \geq \frac{e(N)}{|N|}, \quad \forall \emptyset \neq S \subset N.$$

Because $E = e(S) \cup e(S^c) \cup e(S, S^c)$, for this inequality to hold it must be that:

$$\frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}, \quad \forall \emptyset \neq S^c \subset N.$$

As this is true for all nonempty $S^c \subset N$, we are free to drop the complement superscripts. With Proposition 2 in hand, Proposition 3 obtains.

H Matlab code for Example 2

```
function [Q,W] = GLOBALNET_LIMIT(E,V,phi)

%This program solves for unique q^* in the limit (Theorem 1).

%inputs:
%E (nxn adjacency matrix),
%V (nx1 vector of values v_i)
%phi (network scale factor)

%outputs:
%Q (cutoffs),
%W (limit weighting matrix)

n = length(E);% number of agents
T = sum(V);
e = .5*sum(sum(E));
fun = @(x)gap(x,E,V,phi);
```

```
X0 = ones(n)*.5;% initial point
%options = optimoptions('fminunc','MaxFunctionEvaluations',200000,'StepTolerance',1e-11,'OptimalityTolerance',1e-11,'FunctionTolerance',1e-11);
X = fmincon(fun,X0,[],[],[],[],zeros(n),ones(n),[]);
W = triu(X,1)+(triu(ones(n)-X,1))';
W = E.*W;
Q = V+phi*diag(E*W');

end


%%%%%%%%%%%%%%%%%
% sub-functions %
%%%%%%%%%%%%%%%%%

function value = gap(X,E,V,phi)

n = length(E);
W = triu(X,1)+(triu(ones(n)-X,1))'; %ensures w_ij+w_ji=1 for each i,j.
T = sum(V);
L = T/n*ones(n,1) - (V+phi*diag(E*W'));
value = L'*L; %Euclidean norm

end
```